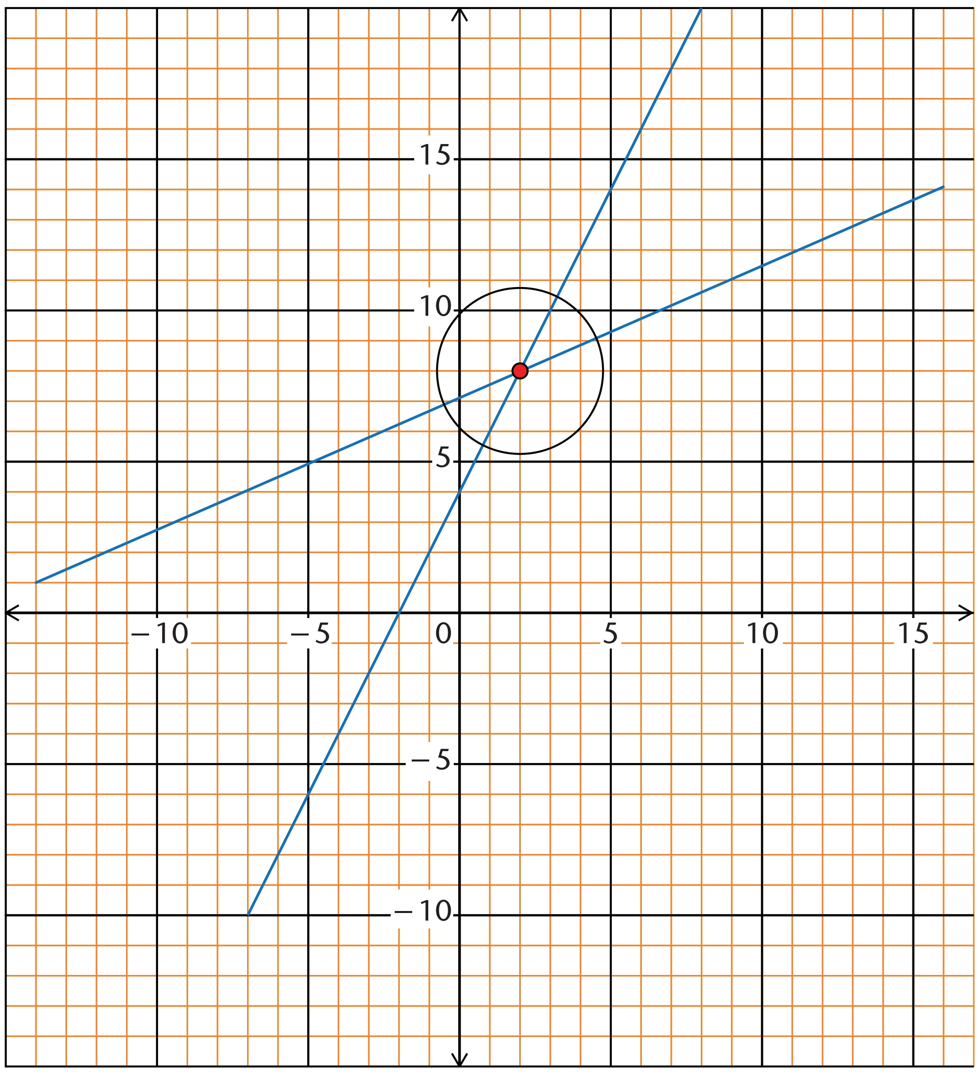
Chapter 3
Equations

You have already solved equations by inspection and inverse operations in the first term. In this chapter you will first revise this work. Then you will work with equations which contain product expressions, like $2x(x - 2)$ and $(x - 5)(x + 3)$. You will learn new methods to solve these equations, based on the fact that if the product of two expressions (or numbers) equals zero, one or both of the expressions or numbers must be zero. You will use factorisation to write equations in the form $pq = 0$ so that you can solve them.


15
10
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3 Equations

3.1 Introduction

SOLUTION BY INSPECTION

1. Complete the following table. Substitute the given x-values into the equation until you find the value that makes the equation true.

You can read the solutions of an equation from a table.

	Equation	LHS if $x = 4$	Is LHS = RHS ?	LHS if $x = 5$	Is LHS = RHS ?	LHS if $x = 6$	Is LHS = RHS ?	Correct solution
(a)	$3x - 4 = 11$							$x =$
(b)	$2x + 7 = 19$							$x =$
(c)	$13 - 5x = -7$							$x =$

(LHS = Left-hand side and RHS = Right-hand side)

2. In the following table, you are given equations with their solutions. Insert + or – or = signs between each term to make the equations true for the solution given.

The "searching" for the solution of an equation is referred to as solving the equation by **inspection**.

	Equation	Solution
(a)	$2x \quad 7 = 15$	$x = 4$
(b)	$3 \quad 2x = 11$	$x = -4$
(c)	$-x \quad 7 = 3$	$x = 4$
(d)	$28 \quad 5x = 3$	$x = 5$

Statements like $21 - x = 2x + 3$ and $(x - 3)(x - 5) = 0$, which are true for only some values of x are called **equations**.

A statement like $2(x + 3) = 2x + 6$, which is true for all values of x you can think of, is called an **identity**.

A statement like $2(x + 3) = 2x + 3$, where there are no values of x for which it is true, is called an **impossibility**.

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SOLVING EQUATIONS THROUGH INVERSE OPERATIONS

In this section you are going to explore a different way of solving equations.

1. Complete the following calculations.
   (a) $3 - 3$ (b) $-9\,765 + 9\,765$
   ....
   (c) $-a + a$ (d) $13a - 13a$
   ....
2. What do you notice?
   ....
3. Complete the following calculations.
   (a) $3 \div 3$ (b) $3 \times \frac{1}{3}$
   ....
   (c) $\frac{1}{x} \times x$ (d) $\frac{x}{3} \times \frac{3}{x}$
   ....
4. What do you notice?
   ....

We can start with a solution as an equation and then apply some operations to it to turn it into an equivalent but more complicated equation.

Two equations are **equivalent** if they have the same solution.

Building an equation

Action on both sides	Equivalent equations
Solution (1)	$x = 3$
$\times 3$	$3x = 9$
$+ 2$	$3x + 2 = 11$
Solution (2)	$x = -9$
$\times 2$	$2x = -18$
$+ 6$	$2x + 6 = -12$
$+ x$	$3x + 6 = x - 12$
factorise	$3(x + 2) = x - 12$

Solving an equation

Action on both sides	Equivalent equations
Equation (1)	$3x + 2 = 11$
$- 2$	$3x = 9$
$\div 3$	$x = 3$
Equation (2)	$3(x + 2) = x - 12$
remove brackets	$3x + 6 = x - 12$
$- x$	$2x + 6 = -12$
$- 6$	$2x = -18$
$\div 2$	$x = -9$

Building an equation	
Action on both sides	Equivalent equations
Solution (3)	$x = 1$
$\times -1$	$-x = -1$
$+3$	$-x + 3 = 2$
$+2x$	$+x + 3 = 2 + 2x$
$\div 2$	$\frac{(x+3)}{2} = 1 + x$

Solving an equation	
Action on both sides	Equivalent equations
Equation (3)	$\frac{(x+3)}{2} = 1 + x$
$\times 2$	$x + 3 = 2 + 2x$
$-2x$	$-x + 3 = 2$
-3	$-x = -1$
$\div -1$	$x = 1$

Try making up your own equations and then solving them. Did you get the "solution" that you started with?

> When you solve an equation, you actually reverse the making of the equation.

5. Solve for x:

(a) $2(x + 4) + 9 = 15$

....

....

....

....

(b) $5(x - 2) = 7(2 - x)$

....

....

....

....

(c) $\frac{2x}{3} - 2 = 12$

....

....

....

....

(d) $\frac{3y-3}{2} + \frac{5}{2} = \frac{5y}{3}$

....

....

....

....

Up to now you have only dealt with equations of the **first degree**. That means they contained only *first powers* of the unknown (x), for example $3x - 2 = 5x + 7$. In the following sections you will solve equations of the **second degree**, where the expressions contain second powers. This is an equation of the second degree:

$x^2 + 1 = x + 13$.

When the expression part of the equation is written as the product of a monomial and a binomial, e.g. $x(x - 2) = 0$; or the product of two binomials, e.g. $(x - 2)(x + 3) = 0$ the result is also an equation of the second degree.

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3.2 Solving by factorisation (Part 1)

DEVELOPING A STRATEGY: MULTIPLYING BY ZERO

1. Can you find two numbers x and y so that if you multiply them the answer is 0, i.e. $xy = 0$?

....

....

....

Each part of a product is called a **factor** of the expression.
If $c = ab$, then a and b are factors of c.
If $x^2 + 5x + 6 = (x + 2)(x + 3)$, then $x + 2$ and $x + 3$ are factors of $x^2 + 5x + 6$.

2. Complete the following table:

	Equation	Factors	Product	First possible solution	Second possible solution
Example	$x(x - 2) = 0$	x and $(x - 2)$	0	$x = 0$	$x - 2 = 0$ $x = 2$
(a)	$x(x + 5) = 0$	....	....	....	....
(b)	$2x(3x - 12) = 0$	....	....	....	....
(c)	$0 = (x + 2)(x - 2)$	....	....	....	....

You can rewrite an equation so that it is in the form *expression* = 0; for example you can write $x^2 - 2x = 3x + 6$ as $x^2 - 5x - 6 = 0$.

You can factorise $x^2 - 5x - 6$ and then use the zero-product property to solve the equation.

$x^2 - 5x - 6 = 0$

$(x - 6)(x + 1) = 0$

$x = 6$ or $x = -1$

Zero-Product Property
If: $a \times b = 0$
Then: $a = 0$ or
$b = 0$ or
$a = 0$ and $b = 0$

In a later section you will solve equations like the above example. You have to write the equation in the form, *expression* = 0, then factorise the left-hand side and then use the zero-product property.

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TAKING OUT THE HIGHEST COMMON FACTOR

The process of writing a sum expression (polynomial) as a product (monomial) is called **factorisation**. This is the inverse of **expansion**.

Look at the expression $2x^2 - 6x$.

$2x$ is a factor of both terms, therefore it is a factor of $2x^2 - 6x$.

By division we get $\frac{2x^2 - 6x}{2x} = x - 3$.

Hence $2x^2 - 6x = 2x(x - 3)$.

It is unnecessary to write out the division step of this method. After finding the common factor, we write down the product form directly.
$2x^2 - 6x = 2x(\qquad\qquad)$

Determine the values of x which will make the following statements true:

1. $x^2 = -3x$

....
....
....
....
....
....
....

2. $x^2 + 2x^2 = 6x$

....
....
....
....
....
....
....

3. $\frac{6x}{3} + x = -4x^2$

....
....
....
....
....
....
....
....

4. $x = x(2 - x)$

....
....
....
....
....
....
....
....

3.3 Solving by factorisation (Part 2)

SOLVING BY FACTORISING TRINOMIALS

The product of the first terms of the factors must be equal to the x^2 term of the trinomial. The product of the last terms of the factors must be equal to the last term (the constant term) of the trinomial.

$$x^2 + 5x + 6 \quad = \quad (x + 2)(x + 3)$$

Meaning: $x.x = x^2$

Meaning: $2.3 = 6$

The sum of the inner and outer products must be equal to the x term of the trinomial.

$$x^2 + 5x + 6 \quad = \quad (x + 2)(x + 3)$$

Meaning: $(2 + 3)x = 5x$

The factors are of the form: $(x.x) + (a + b)x + (a.b) = (x + a)(x + b)$.

Determine the values of x which will make the following statements true.
Remember to write the equation in the form *expression* = 0 so that you can use the zero-product property.

1. $x^2 + 9x = -14$
2. $x^2 + 3x = 18$
3. $x^2 - 18x = -17$
4. $x^2 + 30 = 11x$
5. $x^2 = 13x + 30$
6. $x^2 + 7x = 30$

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SOLVING BY FACTORISING THE DIFFERENCE BETWEEN TWO SQUARES

Remember from the previous chapter:

If p and q are perfect squares, also "algebraic squares", then:

$$p - q = (\sqrt{p} + \sqrt{q})(\sqrt{p} - \sqrt{q})$$

$$9x^4 - 4y^2 = (\sqrt{9x^4} + \sqrt{4y^2})(\sqrt{9x^4} - \sqrt{4y^2})$$
$$= (3x^2 + 2y)(3x^2 - 2y)$$

An expression of the form $a^2 - b^2$ is called the **difference between two squares**.

To factorise a difference between squares, we use the identity: $a^2 - b^2 = (a + b)(a - b)$ where a and b represent numbers or algebraic expressions.

Determine the values of the unknown (x or a or n, etc.) which will make the following statements true.

Remember to write the equation in the form *expression* = 0 so that you can use the zero-product property.

1. $x^2 = 4$
2. $x^2 = 16$
3. $4a^2 = 9$
4. $81 = 9n^2$
5. $25x^2 = 36$
6. $121x^2 = 144$
7. $16p^2 = 49$
8. $64a^2 = 25$

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3.4 Solving by factorisation (Part 3)

SOLVING BY USING PROPERTIES OF EXPONENTS

1. Write the following numbers as the product of their prime factors.

   (a) 128 (b) 243

   ....

   (c) 125 (d) 2 401

   ....

All numbers can be written as the product of their prime factors:
16 = 4 × 4 = 2 × 2 × 2 × 2 = $\mathbf{2^4}$
Factorise the number until all the factors are prime numbers.

2. Determine the values of x which will make the following statements true.

   (a) $2^x = 2^7$ (b) $3^x = 3^5$

   ....

   (c) $5^x = 5^3$ (d) $7^x = 7^4$

   ....

If the base of the LHS is the same as the base of the RHS, then the exponent on the LHS must be equal to the exponent on the RHS.
If $a^x = a^y$, then $x = y$.

3. Determine the values of x which will make the following statements true.

   (a) $2^x = 128$ (b) $3^x = 243$

   ....

   ....

   (c) $5^x = 125$ (d) $7^x = 2\ 401$

   ....

   ....

   (e) $2^x + 9 = 25$ (f) $27(3^x) = 3$

   ....

   ....

   ....

In the equation $2^x = 16$, the letter symbol (x) is the exponent. Equations with the letter symbol as an exponent are referred to as **exponential equations**.

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MIXED EXERCISES FOR MORE PRACTICE

Determine the values of the unknown (x or m or b, etc.) which will make the following statements true.

1. $\frac{6x}{3} + x = -4x^2$

2. $x = x(2 - x)$

3. $x^2 + 2x = 15$

4. $m^2 + 4m = 21$

5. $x^2 + 3 = 4x$

6. $b^2 - 16b = -15$

7. $1 = a^2$

8. $25x^2 = 49$

9. $2^x - 25 = -9$

10. $81(3^x) = 3$

3.5 Set up equations to solve problems

THE MATHEMATICAL MODELLING PROCESS

Consider this problem involving a practical situation.

Printing shop A charges 45c per page and R12 for binding a book.

Printing shop B charges 35c per page and R15 for binding a book.

For a book with how many pages will the two shops charge the same?

You can write an equation to describe the problem.

Let the number of pages for which the work costs the same be x. Then

$45x + 1\,200 = 35x + 1\,500$.

The equation represents a mathematical problem that can be solved without necessarily keeping the practical situation in mind. It is called a **mathematical model** of the practical situation.

Now solve the equation.

$$
\begin{aligned}
45x + 1\,200 &= 35x + 1\,500 \\
45x - 35x &= 1\,500 - 1\,200 \\
10x &= 300 \\
x &= 30
\end{aligned}
$$

We describe this as **analysing** the mathematical model, to produce a **mathematical solution**.

We may now ask what the solution to the mathematical problem ("$x = 30$") means in terms of the practical situation. When the equation was set up above, the symbol x was used as a placeholder for the number of pages in a book for which the two shops would charge the same. So – what does the solution tell you?

The mathematical solution may be **interpreted** to establish what it means in the practical situation.

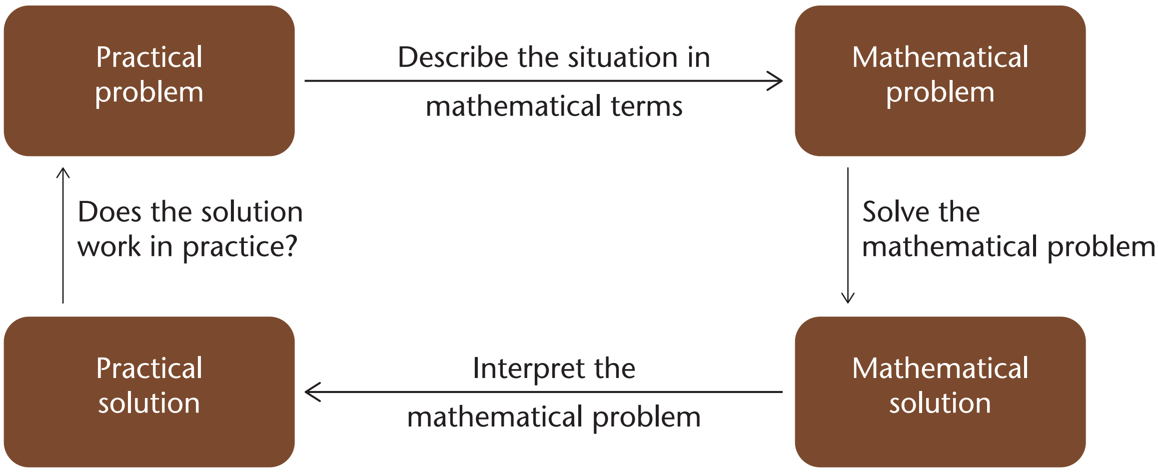
Now check whether the two shops will charge the same for a book with 30 pages.

At shop A 30 pages will cost $30 \times 45\text{c} = 1350\text{c} = \text{R}13{,}50$. Binding is R12, total cost is R25,50.

At shop B 30 pages will cost $30 \times 35\text{c} = 1050\text{c} = \text{R}10{,}50$. Binding is R15, total cost is R25,50.

The solution to the mathematical problem is also a solution to the practical problem.

The mathematical solution should be **tested in the practical situation**, because mistakes may have been made.



When people work like this, we say they do **mathematical modelling**.

PRACTICE YOUR MODELLING SKILLS

For each situation in questions 1 to 3, the mathematical model is outlined and some clues are provided. Fill in the missing information.

1. Louis is 6 years older than Karin and Karin is 4 years older than Heidi. The sum of their ages is 53 years. How old is Heidi?

Model:	Let x be:	Heidi's age
	Then:	Karin's age will be ....
	And:	....
	Hence:	.... = 53
Analysis:	$x + (x + 4) + (x + 10) = 53$	
	....	
Interpretation:	So Heidi is:	

2. The sum of two numbers is 15. Three times the smaller number is 5 more than the larger number. Calculate the two numbers. (**Hint:** Let the smaller number be x.)

Model:	Let x be:	....
	Then:	.... is the larger number
	Hence:	....
Analysis:	....	
Interpretation:	So the smaller number is:	
	And the larger number is:	

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3. The sum of three consecutive even numbers is 108. What are the numbers?
Hint: Consecutive numbers are numbers that follow on each other.
We define an even number as a number of the form $2n$ where n is a counting number.

Model: Let the first number be: ....
Then: ....
....
Hence: ....

Analysis: ....
....

Interpretation: So the first number is: ....
And the second number is: ....
And the third number is: ....

4. Firm A calculates the cost of a job using the formula Cost = 500 + 30t, where t is the number of days it takes to complete the job.
Firm B calculates the cost of the same job using the formula Cost = 260 + 48t, where t is the number of days needed to complete the job.
(a) What would Firm A charge for a job that takes 10 days?
....
....
....
....
(b) How long would Firm B take to complete a job for which their charge is R596?
....
....
....
....
....
(c) Here is a specific job for which firms charge the same and take the same time to complete. How long does this job take?
....
....
....
....
....

3.6 Equations and ordered pairs

WHEN UNKNOWNS BECOME VARIABLES

In the previous sections we dealt with equations which had fixed or limited solutions. They only had one letter symbol, which in this case acted as a placeholder for the value/s which will make the statement true.

Study the equation: $y = 5x + 2$

1. How many letter symbols does the equation have? (List them.)

   ....

2. Is it possible to solve this "equation"?

   ....

3. Complete the table.

x	12	10	20	5	6	–5	–10
$5x + 2$							

FUNCTIONS AS SETS OF ORDERED PAIRS

A specific input number, for example 10, and the output number associated with it (52 in the case of the function described by $y = 5x + 2$) is called an *ordered pair*. Ordered pairs can be represented in a table like the one you completed in question 3 above.

Ordered pairs can also be written in brackets: (input number; output number).

For example the ordered pairs you entered into the table in 3 can be written as
(12; 62), (10; 52), (20; 102), (5; 27), (6; 32),
(–5; –23), (–10; –48)

In the function indicated by $y = 5x + 2$ the letter symbol in the formula (x) represents the **input** or **independent** variable while the other letter symbol (y) represents the **output** or **dependent** variable.

If there is precisely one value of y for each value of x, we say that y **is a function of** x.

1. Complete each table by writing the ordered pairs in brackets below the table, in the table as shown in the example. Then choose two more input numbers and write down two additional ordered pairs that belong to each given function.

   For the function with the rule $y = 4x + 5$

x	–2	0	1	2	5
y	–3	5	9	13	25

(–2; –3), (0; 5), (1; 9), (2; 13), (5; 25), and (10; 45) and (20; 85)

(a) For the function with the rule $y = x^2 + 9$

x	5		0	–3	
y		18			34

(5;34), (3; 18), (0; 9), (–3; 18), (–5; 34), and (. . ; . . .), and (. . ; . . .)

(b) For the function with the rule $y = 3x - 2$

x	5	1	0	–3	
y					–17

(5; 13), (1; 1), (0; –2), (–3; –11), (–5; –17), and (. . . ; . . .) and (. . ; . . .)

(c) For the function with the rule $y = 5x - 4$

x	–5	–3	1	2	
y					21

(–5; –29), (–3; –19), (1; 1), (2; 6), (5; 21), and (. . . ; . . .) and (. . ; . . .)

(d) For the function with the rule $y = 12 - 3x$

x	1	2	3	4	
y					–3

(1; 9), (2; 6), (3; 3), (4; 0), (5; –3), and (. . . ; . .) and (. . ; . . .)

(e) For the function with the rule $y = x^2 + 2$

x	–12	–7	–2	3	
y					102

(–12; 146), (–7; 51), (–2; 6), (3; 11), (10; 102), and (. . . ; . . .) and (. . ; . . .)

(f) For the function with the rule $y = 2^x + 2$

x	0	1	2	3	
y					18

(0; 3), (1; 4), (2; 6), (3; 10), (4; 18) and (. . . ; . . .) and (. . ; . . .)

3. (a) Which ordered pair belongs to both $y = 3x - 2$ and $y = 5x - 4$? . . . .

   (b) Which ordered pair belongs to both $y = 12 - 3x$ and $y = 5x - 4$? . . . .

4. Which ordered pair belongs to both $y = 5x + 7$ and $y = 3x + 25$?

. . . .

. . . .