Chapter 2
Algebraic expressions

$(8 \times 23 - 5) \times 3 \times 23 - (4 \times 23 + 3) \times 6 \times 23$

$= \mathbf{51}$

$3 \times 23 = \mathbf{51}$

$(8 \times 25 - 5) \times 3 \times 25 - (4 \times 25 + 3) \times 6 \times 25$

$= \mathbf{75}$

$3 \times 25 = \mathbf{75}$

$(8 \times 17 - 5) \times 3 \times 17 - (4 \times 17 + 3) \times 6 \times 17$

$= \mathbf{69}$

$3 \times 17 = \mathbf{69}$

$(8 \times 27 - 5) \times 3 \times 27 - (4 \times 27 + 3) \times 6 \times 27$

$= \mathbf{81}$

$3 \times 27 = \mathbf{81}$

2 Algebraic expressions

2.1 Introduction

MANIPULATING EXPRESSIONS

The process of writing a polynomial as a product is called factorisation. This is the inverse of expansion.

$$x^2 + 5x + 6 \underset{\text{expansion}}{\overset{\text{factorisation}}{\rightleftarrows}} (x + 2)(x + 3).$$

A numerical or algebraic expression that requires multiplication as a last step is called a **product**. For example, $12(37 + 63)$, $2x(x - 5)$ and xyz are called products. A product is a monomial.

Each part of a product is called a **factor** of the expression. If $c = ab$, then a and b are factors of c. $x + 2$ and $x + 3$ are the factors of $(x + 2)(x + 3)$. Since $x^2 + 5x + 6 = (x + 2)(x + 3)$, $x + 2$ and $x + 3$ are the factors of $x^2 + 5x + 6$.

1. Calculate the value of each of the following expressions for $x = 12$.

(a) $\dfrac{(x+2)(x+5)}{x+2}$ (b) $\dfrac{(x-3)(x-4)}{x-4}$

....

....

....

....

....

(c) $\dfrac{x(2x+1)}{2x+1}$ (d) $\dfrac{(x+5)(x-5)}{x-5}$

....

....

2. Check whether the following statements are identities by expanding the expressions on the right.

A statement like $2(x + 3) = 2x + 6$, which is true for all values of x you can think of, is called an **identity**.

(a) $x^2 - 9 = (x + 3)(x - 3)$ (b) $x^2 + x - 6 = (x + 3)(x - 2)$

....

(c) $x^2 + 4x + 3 = (x + 3)(x + 1)$ (d) $x^2 + 3x = x(x + 3)$

....

Maths2_Gr9_LB_Book.indb 15 2014/09/08 09:06:10 AM

3. Write down the factors of each of the following expressions.

(a) $x^2 + x - 6$ ....

(b) $x^2 + 3x$ ....

(c) $x^2 + 4x + 3$ ....

(d) $x^2 - 9$ ....

4. Simplify the following quotients (algebraic fractions).

(a) $\dfrac{x^2 - 9}{x + 3}$

....

(b) $\dfrac{x^2 + x - 6}{x + 3}$

....

(c) $\dfrac{x^2 + x - 6}{x - 2}$

....

(d) $\dfrac{x^2 + 4x + 3}{(x + 3)(x + 1)}$

....

5. (a) Suppose you have to find the value of the expression for $x = 15$. Which expression will be the least amount of work? $\dfrac{x^2 - 9}{x + 3}$ or $x - 3$? ....

(b) Are you sure that you will get the same answers for the two expressions?

In the following sections you will learn how to factorise certain types of expressions. The following identities are useful for the purposes of factorisation:

$a(b + c) = ab + ac \qquad (x + a)(x + b) = x^2 + (a + b)x + ab \qquad (a + b)(a - b) = a^2 - b^2$

2.2 Factors of expressions of the form *ab* + *ac*

THE GREATEST COMMON FACTOR

1. (a) Is 5 a factor of 20?

....

(b) Is 5 a factor of 30?

....

(c) Is 5 a factor of 30 + 20?

....

(d) Is 5 a factor of 30 – 20?

....

2. (a) Is a a factor of ab?

....

(b) Is a a factor of ac?

....

(c) Is a a factor of $ab + ac$?

....

(d) Find another factor of $ab + ac$.

....

(e) Now try and simplify: $\dfrac{ab + ac}{a}$ ....

Suppose you have to factorise $4x^3 + 2x^2 - 6x$: It is clear that $2x$ is a factor of every term, hence it is a factor of $4x^3 + 2x^2 - 6x$.

By division we get $\dfrac{4x^3 + 2x^2 - 6x}{2x} = 2x^2 + x - 3$. Hence $4x^3 + 2x^2 - 6x = 2x(2x^2 + x - 3)$.

It is always a good idea to check factorisation by expanding the answer and making sure that the result is equal to the original expression.

3. Complete the table.

For each expression, find:	$3x + 6y$	$4a^3 + 2a$	$5x - 2x^2$	$ax^2 - bx^3$	$12a^2b + 18ab^2$
the factors of the first term	3; x				
the factors of the second term	2; 3; y				
the greatest common factor of the two terms	3				
Write the expression in factor form	$3(x + 2y)$				

4. Study the example and then factorise the expressions that follow.

$$(a - b)x + (b - a)y = (a - b)x - (a - b)y$$
$$= (a - b)(x - y)$$

Note that:
$b - a = -a + b = -(a - b)$

(a) $(a - b)x + a - b$

....

....

(b) $(a - b)x - a + b$

....

....

(c) $(a + b)^2 - (a + b)$

....

....

(d) $(a + b)x - a - b$

....

....

(e) $3x(2x - 3) - (3 - 2x)$

....

....

(f) $(y^2 - 4y) + (3y - 12)$

....

....

SOMETHING IN BETWEEN

1. By completing the tables below you will learn something that will help you to find the factors of expressions of the form $x^2 + (b + c)x + bc$, for example $x^2 + 17x + 30$.

b	1	2	3	5	–1	–2	–3	–5
c	30	15	10	6	–30	–15	–10	–6
$b + c$								
bc								

b	–1	–2	–3	–5	1	2	3	5
c	30	15	10	6	–30	–15	–10	–6
$b + c$								
bc								

Maths2_Gr9_LB_Book.indb 17 2014/09/08 09:06:12 AM

2. For each case below find two numbers x and y so that their product xy is 30 and their sum $x + y$ is the given number.

(a) $xy = 30$ and $x + y = 13$

(b) $xy = 30$ and $x + y = -17$

(c) $xy = 30$ and $x + y = -11$

(d) $xy = 30$ and $x + y = 11$

3. Find x and y in each case.

You may use the tables you completed in question 1 to find the answers to some of these questions.

(a) $xy = -30$ and $x + y = -13$

(b) $xy = 30$ and $x + y = -13$

(c) $xy = -30$ and $x + y = 13$

(d) $xy = -30$ and $x + y = -1$

(e) $xy = -30$ and $x + y = 1$

4. Find x and y in each case.

(a) $xy = 36$ and $x + y = 15$

(b) $xy = 40$ and $x + y = 22$

(c) $xy = 36$ and $x + y = 20$

(d) $xy = -40$ and $x + y = 18$

(e) $xy = 36$ and $x + y = -20$

(f) $xy = -40$ and $x + y = -18$

5. Evaluate each expression for $x = 2$. Also expand each expression.

(a) $(x + 5)(x - 2)$

(b) $(x + 5)(x + 2)$

(c) $(x - 5)(x - 2)$

(d) $(x - 5)(x + 2)$

6. Evaluate each polynomial you formed in question 5 for $x = 2$. Compare the answers with the values of the corresponding product expressions in question 1. In cases where the values differ, you have made a mistake somewhere. Sort out any mistakes completely before you continue with question 7.

7. Expand each product.
   (a) $(x + 3)(x + 8)$ ....
   (b) $(x + 2)(x + 12)$ ....
   (c) $(x + 4)(x + 6)$ ....
   (d) $(x + 1)(x + 24)$ ....
   (e) $(x + 3)(x - 8)$ ....
   (f) $(x + 2)(x - 12)$ ....
   (g) $(x + 4)(x - 6)$ ....
   (h) $(x + 1)(x - 24)$ ....

2.3 Factors of expressions of the form $x^2 + (b + c)x + bc$

The expanded form of a **product of two linear binomials** like $(x + 3)(x + 8)$ or $(x + 3)(x - 8)$ is a **quadratic trinomial** like $x^2 + 11x + 24$ or $x^2 - 5x - 24$ with

- a term in x^2,
- a term in x that is called the **middle term**, which is $+11x$ in $x^2 + 11x + 24$ and $-5x$ in $x^2 - 5x - 24$, and
- a constant term also called the **last term**, which is $+24$ in $x^2 + 11x + 24$, and -24 in $x^2 - 5x - 24$.

To factorise an expression like $x^2 + 5x + 6$ means to reverse the process of expansion. This means that we have to find out which binomials will produce the trinomial when the product of the binomials is expanded, for example:

$$x^2 + 5x + 6 = (\mathbf{?} + \mathbf{?})(\mathbf{?} + \mathbf{?})$$

expansion $(x + 2)(x + 3)$ **factorisation**

$= x^2 + 5x + 6$

TRY TO FIND THE FACTORS

1. Fill in the missing parts of the factors in each of the following cases.
   (a) $(x + 3)(x$ ....$) = x^2 + 9x + 18$
   (b) $(x + 2)(x$ ....$) = x^2 + 11x + 18$
   (c) $(x + 3)(x -$ ....$) = x^2 + 9x - 18$

Maths2_Gr9_LB_Book.indb 19 2014/09/08 09:06:13 AM

(d) $(\ldots + \ldots)(x + 2) = x^2 + 5x + 6$ (e) $x^2 - x - 6$

....

2. Expand each product:
   (a) $(x + p)(x + q)$ ....
   (b) $(x + p)(x - q)$ ....
   (c) $(x - p)(x + q)$ ....
   (d) $(x - p)(x - q)$ ....

The product of the first terms of the factors must be equal to the x^2 term of the trinomial.

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

Meaning: $x \times x = x^2$

Meaning: $2 \times 3 = 6$

The product of the last terms of the factors must be equal to the last term (the constant term) of the trinomial. The sum of the inner and outer products must be equal to the term in x (the middle term) of the trinomial.

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

Meaning: $2x + 3x = (2 + 3)x = 5x$

$(x + a)(x + b) = x \times x + ax + bx + a \times b = x^2 + (a + b)x + ab$

3. Try to factorise the following trinomials.
   (a) $x^2 + 8x + 12$ (b) $x^2 - 8x + 12$

....

PRACTICE MAKES PERFECT

1. Factorise the following trinomials: (Remember to check your answer by expanding the factors to test if you do get the original expression.)

(a) $a^2 + 9a + 14$

(b) $x^2 + 3x - 18$

(c) $x^2 - 18x + 17$

(d) $y^2 + 17y + 30$

(e) $y^2 - 13y - 30$

(f) $y^2 + 7y - 30$

(g) $x^2 + 2x - 15$

(h) $m^2 + 4m - 21$

(i) $x^2 - 6x + 9$

(j) $b^2 + 15b + 56$

(k) $a^2 - 2a - 63$

(l) $a^2 - ab - 30b^2$

(m) $x^2 - 5xy - 24y^2$

(n) $x^2 - 13x + 40$

An alternative method

2. Study the example and then factorise the expressions that follow.

Example: Factorise $ac + bc + bd + ad$

$ac + bc + bd + ad = (ac + bc) + (bd + ad)$ *Order and group terms with common factors*

$= c(a + b) + d(b + a)$ *Take out the common factor*

$= (a + b)(c + d)$ *Write expression as a product*

(a) $px + py + qx + qy$

(b) $9x^3 - 27x^2 + x - 3$

(c) $4a + 4b + 3ap + 3bp$

(d) $a^4 + a^3 + 3a + 3$

....

....

(e) $xy + x + y + 1$

(f) $ac - ad - bc + bd$

....

....

Yet another method

Example 1:	**Example 2:**	**Action:**
$x^2 + 4x + 3$	$x^2 + 3x - 4$	
$= x^2 + x + 3x + 3$	$= x^2 - x + 4x - 4$	*Re-writing middle term as sum of two terms.*
$= (x^2 + x) + (3x + 3)$	$= (x^2 - x) + (4x - 4)$	*Grouping.*
$= x(\mathbf{x + 1}) + 3(\mathbf{x + 1})$	$= x(\mathbf{x - 1}) + 4(\mathbf{x - 1})$	*Taking out the GCF of each group.*
$= (\mathbf{x + 1})(x + 3)$	$= (\mathbf{x - 1})(x + 4)$	*Write it as a product.*

3. Factorise:

(a) $x^2 + 7x + 12$

(b) $x^2 - 7x + 12$

....

The challenge is to re-write the middle term as the sum of two terms in a way that you are able to take out the common factor.

2.4 Factors of expressions of the form $a^2 - b^2$

PRELIMINARY WORK

1. Complete the following table and see if you can notice a pattern (rule) whereby you can predict the answers to the first column's calculations without squaring it:

(a)	$3^2 - 2^2$	$3 + 2$	$3 - 2$	$(3 + 2)(3 - 2)$
(b)	$4^2 - 3^2$	$4 + 3$	$4 - 3$	$(4 + 3)(4 - 3)$
(c)	$6^2 - 4^2$	$6 + 4$	$6 - 4$	$(6 + 4)(6 - 4)$
(d)	$9^2 - 3^2$	$9 + 3$	$9 - 3$	$(9 + 3)(9 - 3)$

Maths2_Gr9_LB_Book.indb 22 2014/09/08 09:06:13 AM

2. Do you notice a pattern (rule) whereby you can predict the answers to such calculations?

....

....

3. Now predict the answers to each of the following without squaring. Check your answers where necessary. Does the rule that you discovered in question 2 also hold for the following cases?

(a) $17^2 - 13^2$ (b) $54^2 - 46^2$ (c) $28^2 - 22^2$

....

....

4. Formulate your rule in symbols:

$a^2 - b^2 =$ ....

5. Can you explain why factors of $a^2 - b^2$ have this form?

....

Stated differently: If p and q are perfect squares, also "algebraic squares", then:

$$p - q = (\sqrt{p} + \sqrt{q})(\sqrt{p} - \sqrt{q})$$

$$9x^4 - 4y^2 = (\sqrt{9x^4} + \sqrt{4y^2})(\sqrt{9x^4} - \sqrt{4y^2})$$

$$= (3x^2 + 2y)(3x^2 - 2y)$$

(Note the operations within the brackets differ.)

An expression of the form $a^2 - b^2$ is called the **difference between two squares**.

To factorise a difference between squares, we use the identity: $a^2 - b^2 = (a + b)(a - b)$ where a and b represent numbers or algebraic expressions.

FACTORISING DIFFERENCE BETWEEN TWO SQUARES EXPRESSIONS

1. Use the skills you learnt in the previous exercises to factorise the following:

(a) $4a^2 - b^2$ (b) $m^2 - 9n^2$

....

(c) $25x^2 - 36y^2$ (d) $121x^2 - 144y^2$

....

(e) $16p^2 - 49q^2$ (f) $64a^2 - 25b^2c^2$

....

(g) $x^2 - 4$ (h) $16x^2 - 36y^2$

....

Maths2_Gr9_LB_Book.indb 23 2014/09/08 09:06:14 AM

Always factorise completely.
Always take out the greatest common factor if there is one.
One is a perfect square: $1 = 1^2$ and $1^m = 1$.
The exponential law: $a^m.a^n = a^{m+n}$.

2. Factorise.

(a) $x^4 - 1$

(b) $16a^4 - 81$

(c) $1 - a^2b^2c^2$

(d) $25x^{10} - 49y^8$

(e) $2x^2 - 18$

(f) $200 - 2b^2$

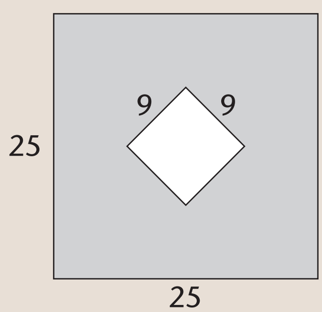
(g) $3xy^2 - 48xa^2$

(h) $5a^4 - 20b^2$

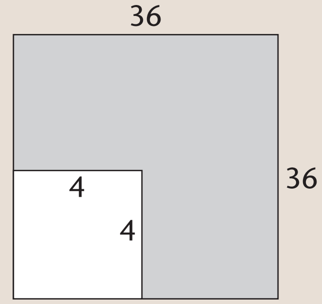
IN EACH CASE CALCULATE THE AREA OF THE SHADED PART.

Use the shortest possible method.

(a)



(b)



THIS IS HOW FACTORISATION CAN MAKE CALCULATION EASY!

2.5 Simplification of algebraic fractions

WORKING WITH ALGEBRAIC FRACTIONS

Liza and Madodo have to determine the value of $\frac{x^2 - 2x - 3}{x - 3}$ for $x = 4{,}6$.

Liza's solution:		Madoda's solution:	
$\frac{x^2 - 2x - 3}{x - 3}$		$\frac{x^2 - 2x - 3}{x - 3}$	
$= \frac{4{,}6^2 - 2(4{,}6) - 3}{4{,}6 - 3}$	Substitute $x = 4{,}6$	$= \frac{(x - 3)(x + 1)}{x - 3}$	Factorise the numerator
$= \frac{21{,}16 - 9{,}2 - 3}{4{,}6 - 3}$		$= x + 1$	Simplify the expression
$= \frac{8{,}96}{1{,}6}$		$= 4{,}6 + 1$	Substitute $x = 4{,}6$
$= 5{,}6$		$= 5{,}6$	

1. Which solution do you prefer? Why?

....

It is useful to manipulate quotient expressions like $\frac{x^2 + 5x + 6}{x + 2}$ into simpler but equivalent sum expressions, like $x + 3$ in this case. It makes substitution and the solving of equations easier.

2. Solve the following problems.

(a) Evaluate $\frac{x^2 + 5x + 6}{x + 2}$ if $x = 23$.

(b) Solve $\frac{x^2 + 5x + 6}{x + 2} = 19$.

....

....

....

....

3. Determine the value of each of the following expressions if $x = 36$. See if you can use the shortest possible method.

(a) $\frac{x^2 - 9}{x + 3}$

(b) $\frac{x^2 + x - 6}{x + 3}$

....

....

....

....

HOW IS IT POSSIBLE THAT 2 = 1?

What went wrong in the following argument?

Let:		$a = b$	(If: $b \neq 0$)
$\times\, a$:	$\Leftrightarrow$	$a^2 = ab$	
$-\, b^2$:	$\Leftrightarrow$	$a^2 - b^2 = ab - b^2$	
Factorise:	$\Leftrightarrow$	$(a + b)(a - b) = b(a - b)$	
$\div\, (a - b)$:	$\Leftrightarrow$	$a + b = b$	
But $a = b$:	$\Leftrightarrow$	$b + b = b$	
Add terms:	$\Leftrightarrow$	$2b = b$	
$\div\, b$:	$\Leftrightarrow$	$2 = 1$	

Explain what went wrong and why it is wrong?

DIVIDING BY ZERO CANNOT BE DONE

1. Complete the following table by evaluating the value of the expression $\frac{x+2}{x-2}$ for the x-values given in the top row:

x	-2	0	2	4
$\frac{x+2}{x-2}$				

2. If $x = 2$ then $\frac{x+2}{x-2}$ will have the value $\frac{4}{0}$. What is the value of $\frac{4}{0}$?

....

3. One way to determine the value of $\frac{4}{0}$, you can set it as $\frac{4}{0} = a$. Then $4 = 0 \times a$. Which values of a will make this statement true?

....

4. What is the result of the calculation of $4 \div 0$ on your calculator? Can you explain the message on your calculator?

....

Division by 0 is not possible. The algebraic fraction $\frac{x+2}{x-2}$ cannot have a value a value if the denominator $(x - 2)$ is equal to 0. We may say the expression $\frac{x+2}{x-2}$ is **undefined** for $x - 2 = 0$ i.e. for $x = 2$. We also say $x = 2$ is an **excluded value** of x for $\frac{x+2}{x-2}$.

DEFINING THE UNDEFINED

1. Is the following statements true? If not, correct the statement.

   (a) $\frac{x}{x} = 1$ for all values of x.

   ....

   ....

   (b) $\frac{x^3}{x^2} = x$ for all values of x.

   ....

   ....

   (c) $\frac{x-3}{x-3} = 1$ for all values of x.

   ....

   ....

   (d) $\frac{x^2+x}{x(x+1)} = 1$ for all values of x.

   ....

   ....

   ....

2. For which values of the variables will each expression be undefined?

   (a) $\frac{7(y+5)}{y+2}$ ....

   (b) $\frac{3x+2}{x+4}$ ....

   (c) $\frac{2x+1}{x^2-1}$ ....

   (d) $\frac{2x^2-1}{(x-2)(x+3)}$ ....

SIMPLIFYING ALGEBRAIC FRACTIONS

To simplify an algebraic fraction that contains a polynomial as numerator or denominator, the polynomial should be factorised first.

To prevent division by zero, the excluded values must be stated.

1. Simplify each of the following algebraic fractions by factorising the numerator and then using the property $\frac{ax}{a} = x$ if $a \neq 0$. Give excluded values.

(a) $\frac{3xy + y^2}{3x + y}$

(b) $\frac{a^2b + ab^2}{a + b}$

....

....

(c) $\frac{3x^2y - 6x^2y^2}{3xy}$

(d) $\frac{10x^4 + 15x^3}{5x^2}$

....

....

2. Simplify each of the following algebraic fractions by factorising the numerator and then using the property $\frac{ax}{a} = x$ if $a \neq 0$. (See if you can factorise the trinomials.)

(a) $\frac{x^2 + 5x + 6}{x + 2}$

(b) $\frac{x^2 + 2x - 8}{x - 2}$

....

....

(c) $\frac{x^2 - 5x - 50}{x + 5}$

(d) $\frac{x^2 - 16x + 15}{x - 15}$

....

....

3. Simplify each of the following algebraic fractions by factorising the numerator and then using the property $\frac{ax}{a} = x$ if $a \neq 0$.

(a) $\frac{x^2 - 4}{x - 2}$

(b) $\frac{4x^2 - 1}{2x + 1}$

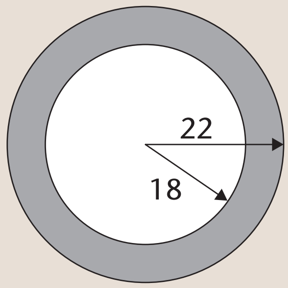
....

....

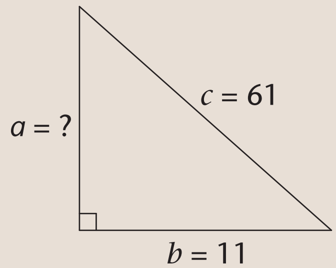
FACTORISATION CAN REDUCE CALCULATIONS

In each case, use the shortest possible method to get to your answer.

(a) Calculate the shaded area.
(Area = πr^2 and use $\pi = 3.142$)



(b) Calculate the length of side a
(Pythagoras: $c^2 = a^2 + b^2$)



THIS IS HOW FACTORISATION CAN SAVE YOU TIME!

MORE PRACTICE

1. Factorise the following expressions completely.

(a) $4a + 6b$

(b) $x^2 + 8x + 7$

(c) $c^2 - 9$

(d) $y^2 - 8y + 15$

(e) $-3ab + b$

(f) $-3a(b - 1) + (b - 1)$

(g) $dfg^2 + d^2g - df^2g$

(h) $x^2 + 6x + 8$

(i) $a^2 + 5a + 6$

(j) $x^2 - 8x - 20$

(k) $x^5y^3 - x^3y^5$

(l) $x^3y - xy^3$

(m) $4 - 4y + y^2$

(n) $3a^2 + 18a - 21$

(o) $6a^2 - 54$

(p) $-a^2 - 11a - 30$

(q) $2a^2 + 10a - 72$

(r) $5x^3 - 15x^2 - 200x$

....

....

(s) $(x + 2)^2 - y^2$

(t) $(x + y)^2 - a^2$

....

(u) $(a^2 - 2a + 1) - b^2$

(v) $1 - (a^2 - 2ab + b^2)$

....

....

(w) $(a - b)x + (b - a)y$

(x) $a(2x - y) + (y - 2x)$

....

....

(y) $2x^2y^{10} - 8x^{10}y^2$

(z) $(a + b)^3 - 4(a + b)$

....

....

(aa) $(a + b)^2 - a - b$

(ab) $(x + y)(a - b) + (-x - y)(b - a)$

....

....

....

2. Simplify each of the following algebraic fractions as far as possible.

(a) $\dfrac{16 - 9x^2}{4 + 3x}$

(b) $\dfrac{25x^2 - 36}{5x^2 + 6x}$

....

....

(c) $\dfrac{x^3 + x^2 - 30x}{x + 6}$

(d) $\dfrac{2x^2 + 5x + 3}{2x + 3}$

....

....

(e) $\dfrac{ab + bc}{abc}$

(f) $\dfrac{pa + pb}{a + b}$

....

....