STRAND 4 → GEOMETRY

HISTORY OF GEOMETRY

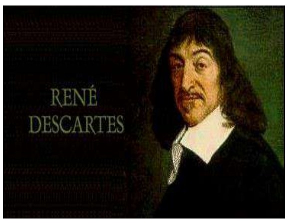
Geometry's was originated about 3,000 BC in ancient Egypt. Egyptians used an early stage of geometry in several ways, including the surveying of land, construction of pyramids, and astronomy. Around 2,900 BC, ancient Egyptians began using their knowledge to construct pyramids with four triangular faces and a square base.

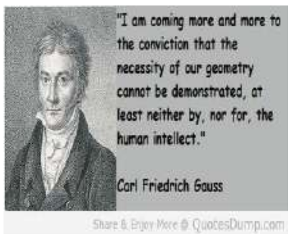
The next great advancement in geometry came from Euclid in 300 BC when he wrote a text titled 'Elements.' In this text, Euclid presented an ideal form in which propositions could be proven through a small set of statements that were accepted as true. In fact, Euclid was able to derive a great portion of planar geometry from just the first five postulates in the 'Elements.' These postulates are listed below:

- *A straight line segment can be drawn joining any two points.*
- *A straight line segment can be drawn joining any two points.*
- *Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.*
- *All right angles are congruent.*
- *If two lines are drawn which intersect a third line in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines must intersect each other on that side if extended infinitely. Euclid's fifth postulate is also known as the parallel postulate.*

The next advancement in the field of geometry occurred in the 17th century when René Descartes discovered coordinate geometry. The creation of coordinate geometry opened the doors to the development of calculus and physics.

In the 19th century, Carl Friedrich Gauss, Nikolai Lobachevsky, and János Bolyai formally discovered non-Euclidean geometry. In this kind of geometry, four of Euclid's first five postulates remained consistent, but the idea that parallel lines do not meet did not stay true. This idea is a driving force behind elliptical geometry and hyperbolic geometry.





Source: http://www.wyzant.com/resources/lessons/math/geometry/introduction/history_of_geometry

TRIGONOMETRY

Trigonometry is the study of the ratios of the sides of triangles. In Trigonometry, Trig refers to triangles and metry means to measure.

4.1 Square and Square Roots

LEARNING OUTCOME

Students should be able to:

- Calculate squares and square roots.

Square

- Is a number multiplied to itself example 3×3.
- In short it is written as 3^2(3 to the power of 2)
- Squaring a negative number always gives a positive answer.

Example 4.1

Find the following squares:

(a) $5^2 = 5 \times 5 = 25.$

(b) $(-5)^2$= -5 × -5 = (- × -) (5 × 5) = 25

(c) - 3^2 = - (3 × 3) = -9

Calculator working

Press 3

Press x^2

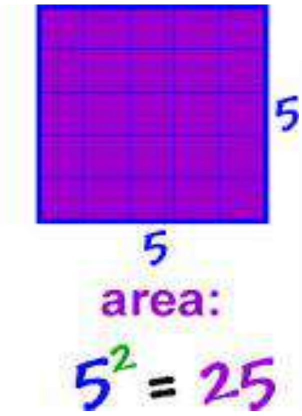
Press =

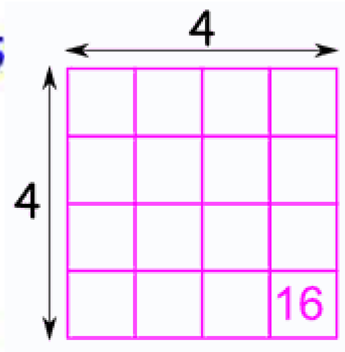
Note :

Always use brackets while squaring a negative number

Perfect Square

Are squares of real numbers example 1, 4, 9, 16, 25, 36, 49, 64, 81……





Square roots ($\sqrt{\ }$ *or* $\sqrt[2]{\ }$)

A square root is the opposite of squaring a number.

Square (3^2)

$3 \leftarrow 9$

Square root ($\sqrt{9}$)

A square root of a number is a value that can be multiplied by it to give the original answer example $\sqrt{49} = 7$ *means that* $7 \times 7 = 49$

Example 4.2

Find $\sqrt{25}$

Solution

Since 25 = 5 x 5

means $\sqrt{25}$ =

We know that 25 = 5 × 5 so $\sqrt{25}$ is 5.

Calculator

Press

Press 25

Press

Exercise 4.1

Find the following square and square roots correct to decimal places.

1. $(2)^2$	2. $(-4)^2$	3. -5^2	4. $\sqrt{16}$
5. $\sqrt{196}$	6. 3.14^2	7. $\sqrt{50}$	8. -7^2
9. $\sqrt{56.25}$	10. $(-8)^2$	11. $\sqrt{4}$	12. $6^2 \times (-3)^2$

4.2 Pythagoras Theorem

Pythagoras Theorem Is a theorem that gives the relationship between the sides of a right - angled triangle.

LEARNING OUTCOMES

BRAIN TEASER

FIND THE SQUARE ROOT OF - 4

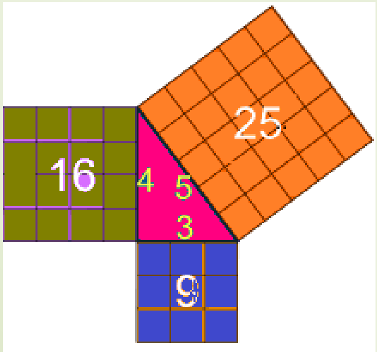
Students should be able to:

- Give the relationships between the sides of a right- angled triangle.
- Use this relationship to find the unknown sides of a right – angled triangle.
- Apply the Pythagoras theorem in real life situation,

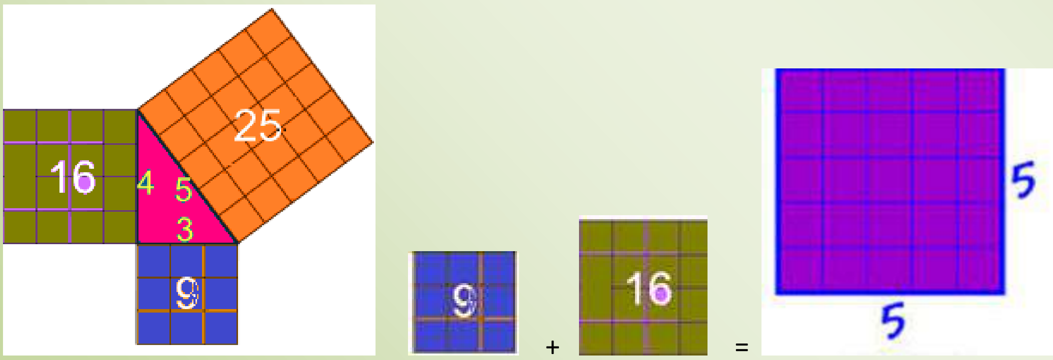
History

Over 2000 years ago there was an amazing discovery made by Pythagoras's about triangles.

"When the triangle has a right angle (90˚) and squares are made on each of the three sides, then the biggest square has the exact same area as the other two squares put together"



The relationship discovered by Pythagoras's is called the Pythagoras's theorem and can be written as $a^2 + b^2 = c^2$.

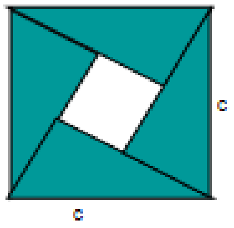


Note:

1.In this case *c* is the longest side of the triangle and is called the hypotenuse, *a* and *b* are the other two sides of the triangle. The hypotenuse is the side opposite the right angle.

2. Given any two sides of the right angled triangle the pythagoras thoerem can be used to find the length of the unknown side.

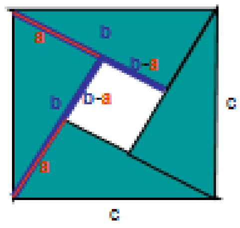
3. The theorem can also be used to determine whether a triangle is a right angled triangle or not.



Area of whole square

$= c * c = c^2$

must be equal



Area of whole square

$= \text{area of 4 green triangles} + \text{area of white square}$

$= 4\left(\frac{1}{2}ab\right) + (b-a)(b-a)$

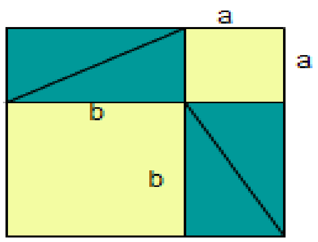
$= 2ab + b(b-a) - a(b-a)$

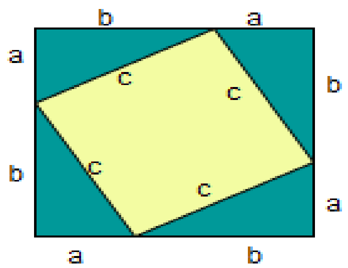
$= 2ab + b^2 - ab - ab + a^2$

$= 2ab + b^2 - 2ab + a^2$

$= b^2 + a^2$

Proof 2





Notice that each square has 4 dark green triangles.
Therefore, the yellow regions must be equal.

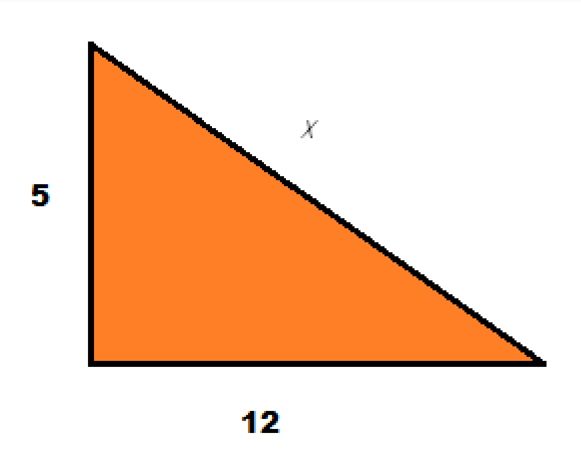
Yellow area $= a^2 + b^2$

Yellow area $= c^2$

Source:
http://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=2&ved=0CCQQFjAB&url=http%3A%2F%2Fwww.math.unl.edu%2F~sdunbar1%2FExperimentationCR%2FLessons%2FGeometry%2FPythagorean%2FPythagoreanTheorem.ppt&ei=bdq9VOauCIXOmwWE9IGIAg&usg=AFQjCNEputeM7MH3CwS9cNOOYj1TmwLRqQ

Example 4.3

(a) Find the unknown sides in the triangles given below.



Solution

$a^2 + b^2 = c^2$

$5^2 + 12^2 = x^2$

$25 + 144 = x^2$

$169 = x^2$

$x^2 = 169$

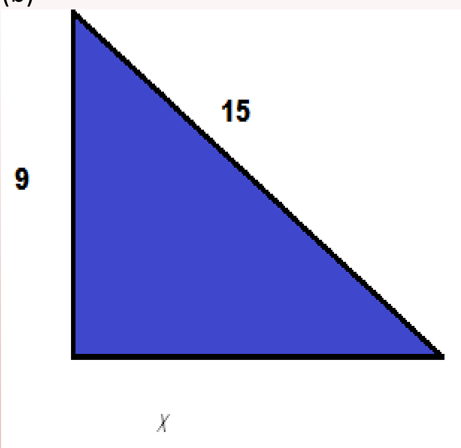
$x = \sqrt{169}$

$x = 13$

Calculate the remaining side

Solution

(b)



$a^2 + b^2 = c^2$

$9^2 + b^2 = 15^2$

$81 + b^2 = 225$ Taking 81 away from both sides gives

$\cancel{81} + b^2 \cancel{- 81} = 225 - 81$

$b^2 = 144$

$b = \sqrt{144}$

$b = 12$

(c) A triangle has lengths 8, 15 and 16. Is it a right angled triangle?

Solution

$a^2 + b^2 = c^2$

$8^2 + 15^2 = 289$

8, 15, 16

c

$a \qquad b$

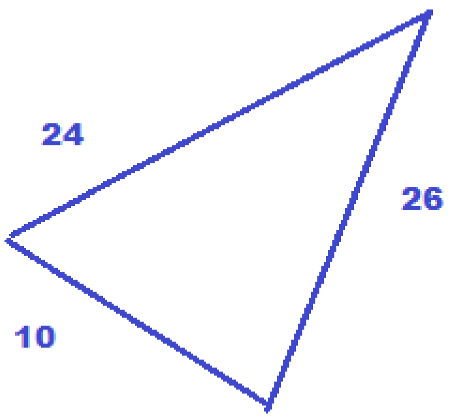
$16^2 = 256$

since $a^2 + b^2$ is not equal to c^2 $(a^2 + b^2 \neq c^2)$ therefore the given triangle is not a right - angled triangle.

Example 4.4

Show that the triangle given below is a right – angled triangle.

(a)



Show that $a^2 + b^2 = c^2$

Note: assign 26 to c since c is the longest side of the triangle.

$a = 10 \quad b = 24 \quad c = 26$

$10^2 + 24^2 = 676$

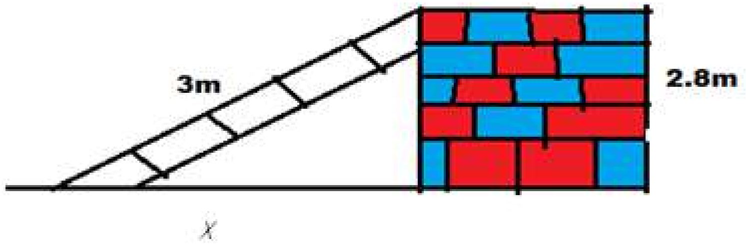
$26^2 = 676$

Since $a^2 + b^2 = c^2$ is *satisfied*

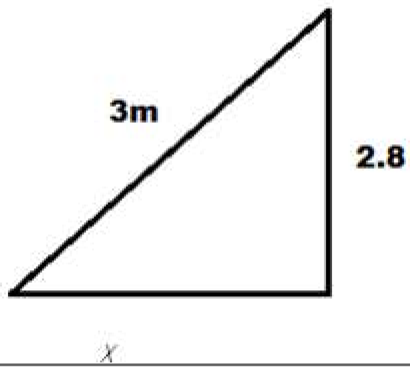
Therefore the given triangle is a right – angled triangle.

Example 4.5

A 3m ladder stands on a horizontal ground and reaches 2.8m up a vertical wall. How far is the foot of the ladder from the base of the wall?



Interpret and illustrate the question in mathematical terms.



Solution

Using the Pythagoras theorem $a^2 + b^2 = c^2$ to calculate the value of x.

Let $a = x$, $c = 3m$ and $b = 2.8m$.

$$a^2 + b^2 = c^2$$

$$x^2 + (2.8)^2 = 3^2$$

$$x^2 + 7.84 = 9$$

$$x^2 + 7.84 - 7.84 = 9 - 7.84$$

$$x^2 = 1.16$$

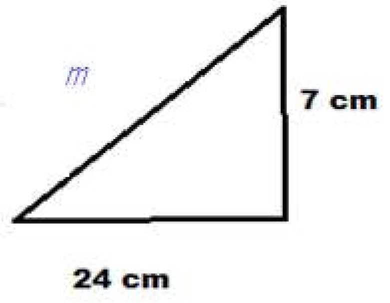
$$x = \sqrt{1.16}$$

$$\boldsymbol{x = 1.08m}$$

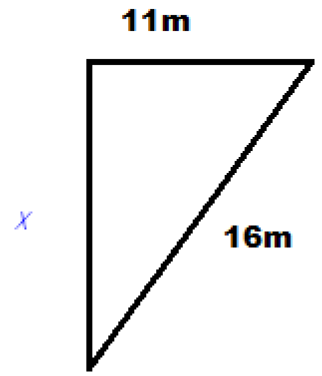
Exercise 4.2

1. Find the missing sides

(a)



(b)



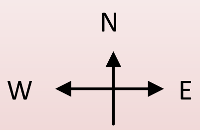
2. The side lengths of various triangles are given. Determine which ones are right angled triangles.

(a) {6, 8, 10}

(b) {3, 5, 6}

(c) { $\sqrt{3}$, $\sqrt{11}$, $\sqrt{8}$}

3. Town B is 8 miles north and 17 miles west of town A. How far are the two towns?



A rectangular field is 125 m long and the length of one diagonal of the field is 150m. What is is the width of the field.

A 8m ladder is leaned against the side of a wall. How high does the ladder reach if its base is is 3m away from the building.

4. A rectangular field is 125 m long and the length of one diagonal of the field is 150m. What is the width of the field?

5. A 8m ladder is leaned against the side of a wall. How high does the ladder reach i reach if its base is 3m away from the building?

6. Linda is mountain climbing with Allie and has just climbed a 16-metre vertical roc rock face. Allie is standing 12 metres away from the bottom of the cliff, looking up a up at Linda. How far away are Linda and Allie?

4.3 Trigonometric Functions

LEARNING OUTCOMES

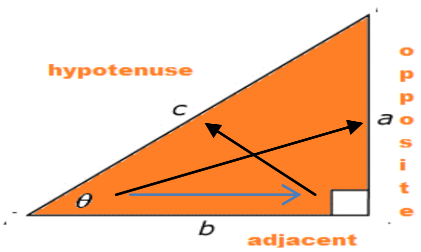
Students should be able to:

- Describing the three basic trigonometric functions.
- Calculating sine, cosine and tangent values of theta and vice versa.
- Use SOH, CAH, TOA to find unknown side and angle of right angle triangle.

4.3.1 Naming the Sides of a Right Angled Triangle

- In trigonometry the Greek letter Ө (theta) is used as the name of an angle.
- Using Ө the sides of the triangle can be named.

For example



The sides of the triangle can be labelled as a, b and c. Side a is opposite of Ө, side b adjacent to Ө and side c is the hypotenuse (the longest side) opposite the right angle. From this, 3 functions can be introduced.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

NOTE : Sin Ө ÷ Cos Ө = Tan Ө

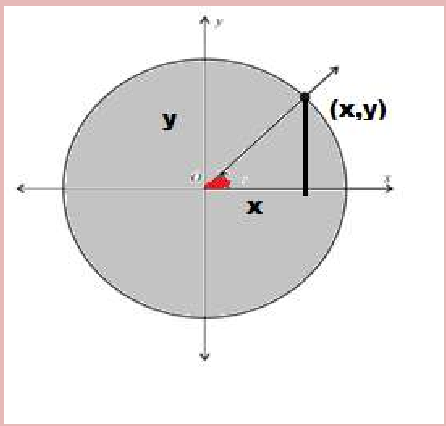
In short SOH, CAH, TOA can be used.

Example 4.6

Given below is a unit circle. A unit circle is circle of radius 1 unit. Θ is the angle marked in red

Questions

(i) What is the length of the hypotenuse?
(ii) Find Cos Θ
(iii) Find Sin Θθ
(iv) Find Tan Θ



(i) The size of the hypotenuse is I since it is the radius of a unit circle.
(ii) Cos Θ = $\frac{a}{h}$, so Cos Θ = $x/1$ = x
(iii) Sin Θ = $\frac{o}{h}$ = $\frac{y}{1}$ = y
(iv) Tan θ = $\frac{o}{a}$ = $\frac{y}{x}$

Sin θ and Cos θ are the x and y coordinates of the point (1, 0) as it is rotated by θ degrees about the origin

4.3.2 Using A Calculator

- A calculator can be used to find the values of the trigonometric ratios sin, cos and tan for the given angle where the angle is measured in degrees.
- In the same way if the trigonometric ratios are given then the angle can also be found using the inverse function.

Example 4.7

Example One: Find Sin 30°

Solution : Using the calculator

Press sin

Press 30

Press =

sin 30 = 0.5.

Example Two

Find sin ϴ = 0.66

Solution

Press shift

Press sin

Press 0.66 =

sin^{-1} (0.66) = 41.30 (2 dp)

Exercise 4.3

1.Evaluate the following

(a) Sin 60°

(b) Tan 34°

(c) Cos 124°

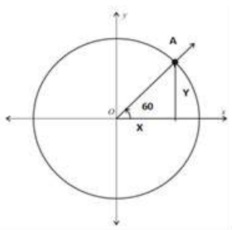
2. Find the value of ϴ

(a) Cos θ = 0.54

(b) Sin θ = 0.76

(c) Tan θ = 0.45

3. The diagram given below shows a unit circle.



(i) Find the lengths of x and y.

(ii) Write coordinates of A.

(iii) Use your calculator to find Sin 60 and Cos 60. Is it the same as the x and y coordinates of point A?

(iv) Find tan 60 and compare it with sin 60

4.3.3 Applications of SOH, CAH, TOA

LEARNING OUTCOMES

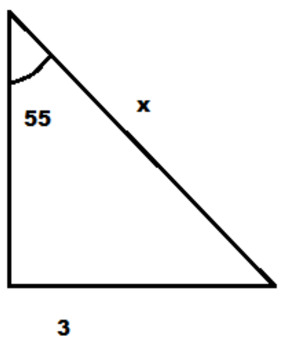
Students should be able to use SOHCAHTOA to calculate the:

- Unknown side of a right angle triangle
- Unknown angle of a right angle triangle

SOH CAH TOA CAN BE USED TO FIND THE UNKNOWN SIDE AND ANGLES OF A RIGHT ANGLED TRIANGLE

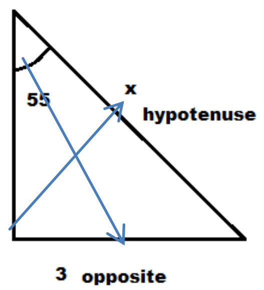
Example 4.8

Find the length of the side marked x in the right - angled triangle given below



Step one

Identify the given sides of the triangle.



Step two

Determine the trig function to be used

SOH CAH or TOA

Since O and H are given we use SOH

$$\sin \Theta = \frac{O}{h}$$

Step three

$\Theta = 55^{0}$ O = 3m H - x

$$\sin 55 = \frac{3}{x}$$

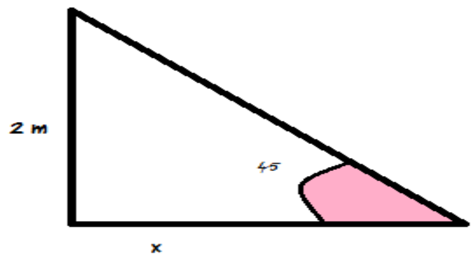
x sin 55° = 3 (multyplying x on both sides)

$x = 3/\sin 55$ (dividing by sin 55 on both sides)

x = 3.66m (2 dp)

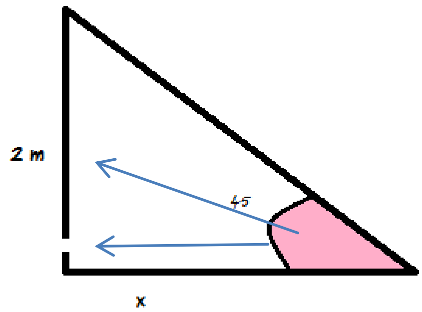
Example 4.9

Find the side marked x in the right - angled triangle given below



Step one

Identify the sides and angles given.



adjacent

Step two

Identify the trig function SOH CAH or TOA

Since the opposite and adjacent are given

TOA is used.

Step three

$\tan\theta = \frac{o}{a}$ $\quad o = 2 \quad a = x \quad \theta = 45°$

$\tan 45 = \frac{2}{x}$

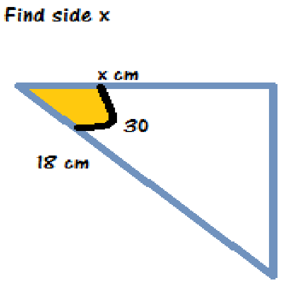
$x \tan 45 = 2$ (multiplying by x on both sides)

$x = 2/\tan 45$

$\underline{x = 2\ m}$

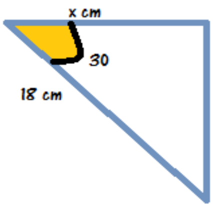
Example 4.10

A right - angled triangle is given below



Step one

Identify the sides and the angles.



Hypotenuse -18cm

Adjacent – x cm   θ= 30^0

Step two

Determine the trig function to use

Since a and h are given

CAH is used.

$Cos\ \theta = \frac{a}{h}$ $Cos\ 30 = \frac{x}{18}$

Step three

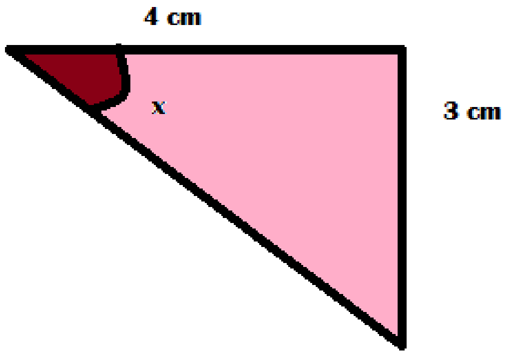
$Cos\ 30 = \frac{x}{18}$

$18 \cos 30 = x\ (multiplying\ 18\ by\ both\ sides)$

$\boldsymbol{x = 15.59\ cm}$

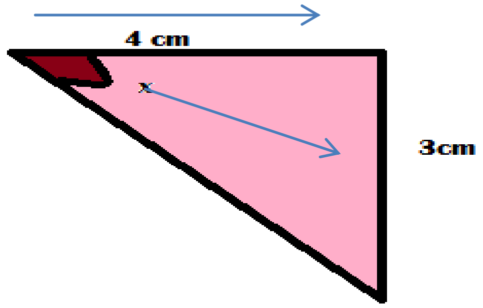
Example 4.11

For the right - angled triangle given, find angle x



Step one

Identify the sides and the angles.



θ = x, opposite – 3cm, adjacent – 4cm

Step two

Determine the trig function to use

Since o and a are given

TOA is used.

$\tan \theta = \frac{o}{a}$ $\tan x = \frac{3}{4}$

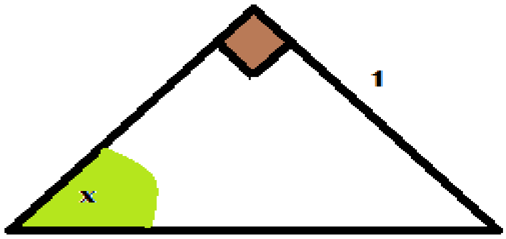
Since the angle is the unknown we take $\tan^{-1}$ on both sides. $\tan^{-1} 3/4$ = __36.87°__

Note: while solving this types of problems ensure that the calculator is in the degree mode.

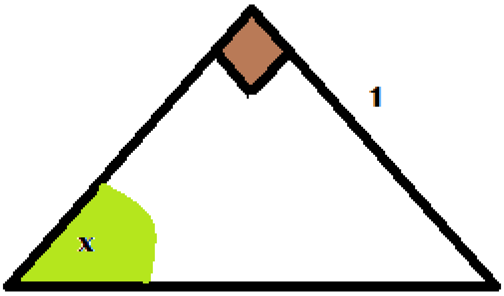
Example 4.12

The right angled triangle below is also an isosceles triangle. Find angle x

Step one



Identify the sides and the angles



Opposite – 1 $\theta = x$ hypotenuse - $\sqrt{2}$

Step two

Identify the trig function to use from

SOH CAH TOA.

Since o and h are given

SOH is used

$$Sin\ \theta = \frac{o}{h}$$

Step three

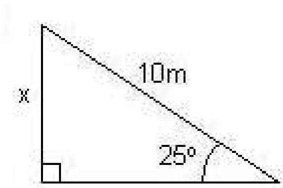
$\sin x = 1/\sqrt{2}$

$\sin^{-1}\frac{1}{\sqrt{2}} = 45^{0}$

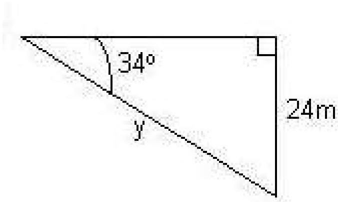
x = **<u>45°</u>**

Exercise 4.4

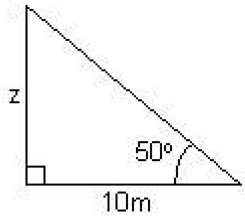
1. Find the unknown sides and angles either using SOH CAH or TOA.
   (a)



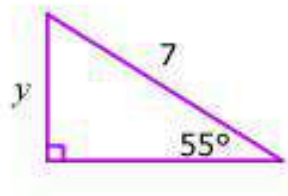
(b)



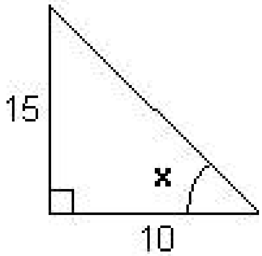
(c)



(e)



(d)



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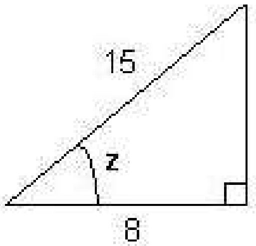
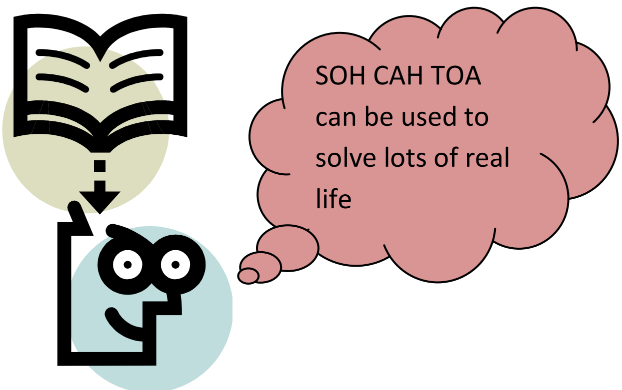


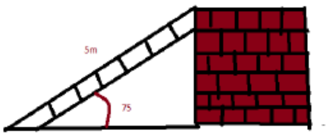
Trignometric Word Problems



Example 4.13

A ladder leaning against a wall makes 75^{0} angle with the ground. If the ladder is 5m tall, how far is the base of the ladder from the wall of the house?

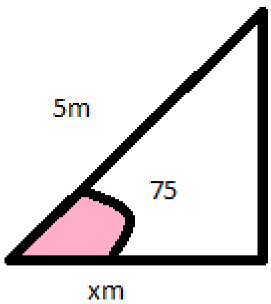
A diagramatic representation of the problem is given below.



Solution

Step one

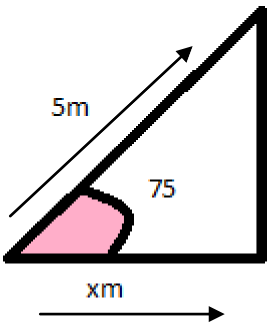
Take out the triangle right angled triangle formed



$\theta = 75^{0}$ adjacent – x hypotenuse- 5m

Step two

Identify the sides and angles given



Step three

Identify the trig function to use

Since a and h are given

CAH is used

$\cos \Theta = a/h$

Step four

$\Theta = 75^{0}$, a = x m, h = 5m

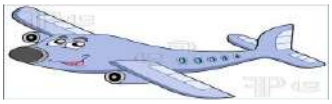
$\cos 75 = x/5$

$5 \cos 75 = \mathbf{1.29\ m}$

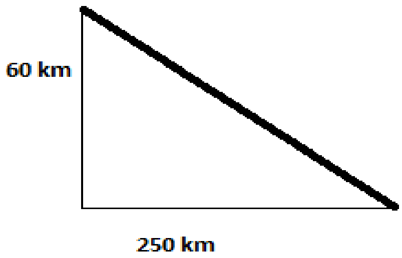
Hence the ladder is 1.29m away from the wall.

Example 4.14

An aeroplane flies 250 km west, but the wind blows it 60 km north



What is the compass bearing of the plane from the starting point?



Step one

compute angle x first

θ= x   opposite – 60 km adjacent – 250km

Use TOA since opposite and adjacent are given

Step two

$\tan x = \frac{60}{250}$

$tan^{-1}\left(\frac{60}{250}\right) = 13.5$

Step three

Total bearing = 90 +180+x

90+180+13.5 = 283.5

Exercise 4.5

1. Determine the correct formula for the sine ratio

A. $\text{Sin x} = \frac{\text{hypotenuse}}{\text{opposite}}$

B. $\text{Sin x} = \frac{\text{opposite}}{\text{adjasent}}$

C. $\text{Sin x} = \frac{\text{opposite}}{\text{hypotenuse}}$

D. $\text{Sin x} = \frac{\text{adjacent}}{\text{hypotenuse}}$

2. In $\triangle$ ABC, AB = 10 cm, $\angle B = 90^o$, and $\angle C = 60^o$. Determine the length of BC, to the nearest centimeter

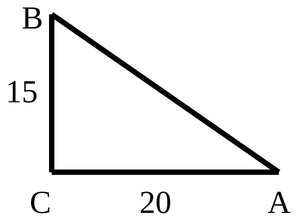
A. 5 cm

B. 6 cm

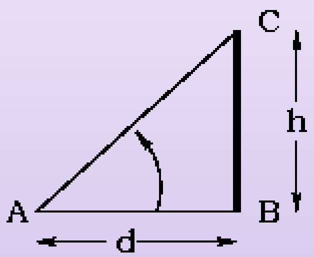
C. 6 cm

D. 8 cm

3. In right triangle ABC, leg BC=15 and leg AC=20. Find angle A



4. A man is walking along a straight road. He notices the top of a tower subtending an angle $A = 60^o$ with the ground at the point where he is standing. If the height of the tower is h = 35 m, then what is the distance (in meters) of the man from the tower?



5. A little boy who is 1.5m tall is flying a kite. The string of the kite makes an angle of 30^o with the ground. If the height of the kite is h = 12 m, find the length (in meters) of the string that the boy has used.

4.3.4 Graphs of Trigonometric Functions

LEARNING OUTCOMES

Students should be able to:

- Sketch the trigonometric graphs.
- Identify the graphs of sine, cosine and tangent

Applications of the Trignometric Graphs

Music is an integral part of life of most people. Although the kind of music they prefer differs.

All music is the effect of the sound waves on the ears.

No matter what vibrating object is causing the sound wave the frequency of the wave (that is the number of waves per second) creates a sensation that is called the pitch of sound.

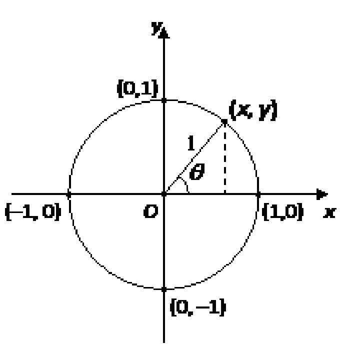
Sound is just one of the many physical entities that are transmitted by waves. Light , radio, television, X-rays, and microwaves are the others. The trigonometric Graphs that we study in this topic provide the mathematical basis for the study of waves.

The Sine Graph

The Sine Graph can be plotted by using the unit circle .

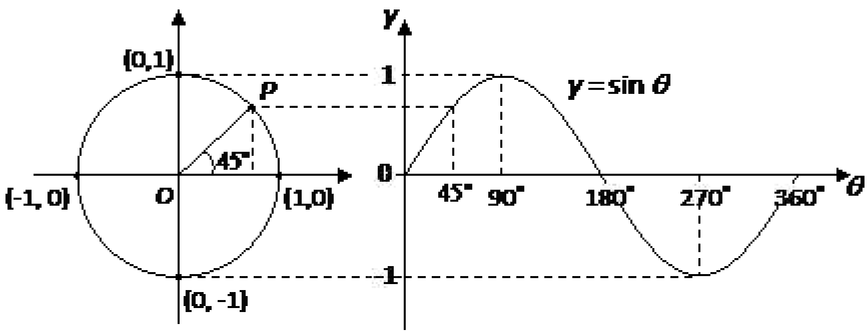
Recall:

A unit circle is a circle of radius one unit with its centre at the origin.



- ***y* = sin θ is known as the sine function.**
- Using the unit circle, we can plot the values of *y* against the corresponding values of θ.

The **graph of $y = \sin\theta$, for $0^\circ \leq \theta \leq 360^\circ$** obtained is as shown:



Properties of the Sine function:

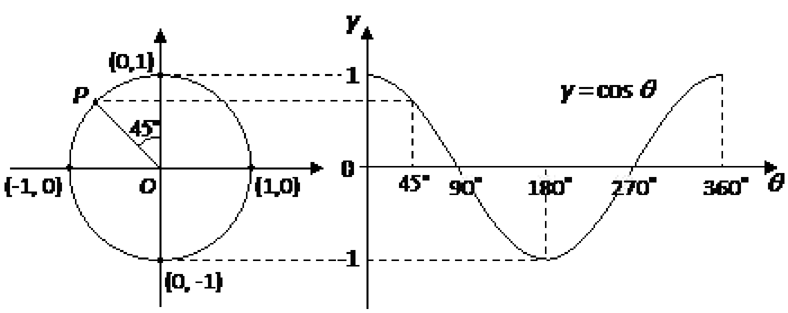
- The sine function forms a wave that starts from the origin
- sin θ= 0 when $\theta = 0^\circ, 180^\circ, 360^\circ$.

Maximum value of sin θ is 1 when $\theta = 90^\circ$. Minimum value of sin θ is –1 when $\theta = 270^\circ$. So, the range of values of sin θ is $-1 \leq \sin\theta \leq 1$.

The Cosine Graph

- ***y* = Cos θ is known as the cosine function.**
- Using the unit circle, the values of *y* against the corresponding values of **θ** *can be plotted.*

The **graph of $y = \cos\theta$, for $0^\circ \leq \theta \leq 360^\circ$** obtained is as shown:

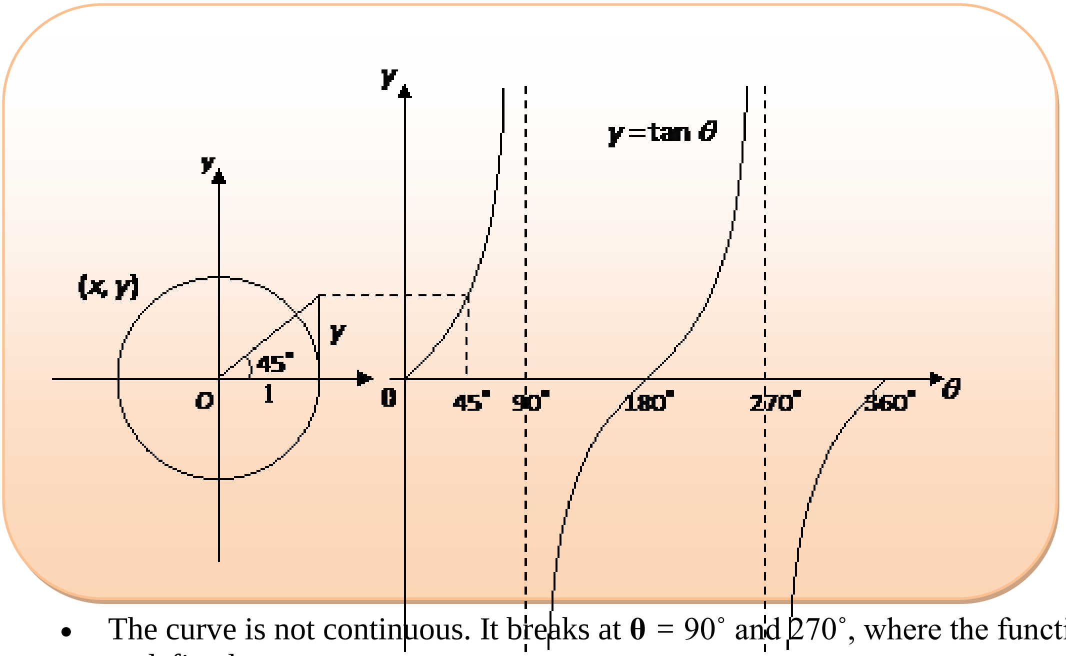


Properties of the Cosine function:

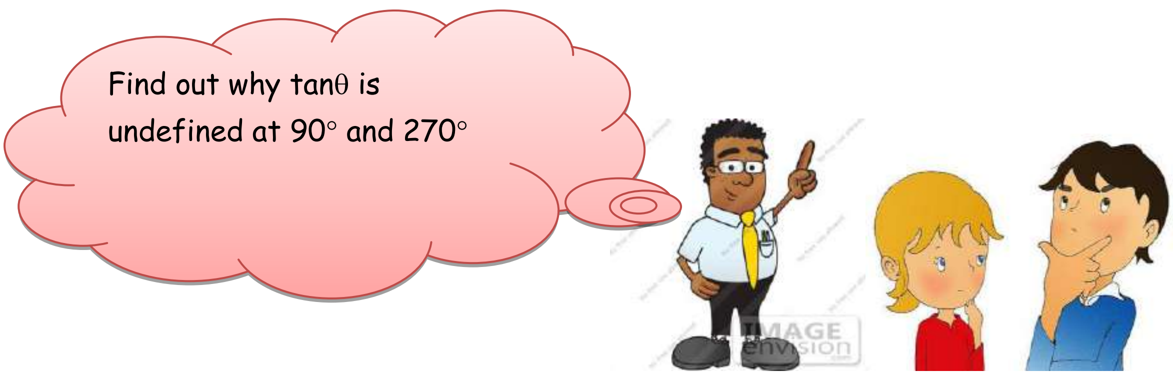
- The cosine function forms a wave that starts from the point (0,1)
- cos **θ** = 0 when **θ** = 90˚, 270˚.
- Maximum value of cos **θ** is 1 when **θ** = 0˚, 360˚. Minimum value of cos **θ** is –1 when **θ** = 180˚. So, the range of values of cos **θ** is $-1 \leq \cos\theta \leq 1$.

The Tangent Graph

- **y= tan θ is known as the tangent function.**
- Using the unit circle, values of *y* against the corresponding values of **θ** can be plotted. The **graph of *y* = tan θ, for 0˚ ≤ θ ≤ 360˚** obtained is as shown:

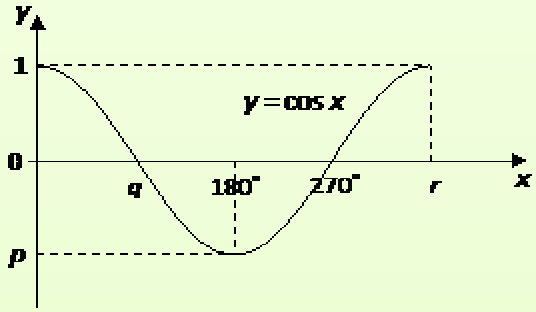


- The curve is not continuous. It breaks at **θ** = 90˚ and 270˚, where the function is undefined.
- tan **θ**= 0 when **θ**= 0˚, 180˚, 360˚. tan **θ** = 1 when **θ**= 45˚ and 225˚.
- tan **θ** = –1 when **θ**= 135˚ and 315˚.
- tan **θ** does not have any maximum or minimum values. The range of values of tan **θ** is $-\infty < \tan\theta < \infty$



Exercise 4.6

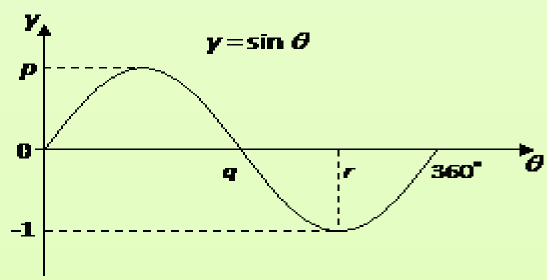
1. The diagram shows a graph of $y = \cos x$ for$0^\circ \leq x \ \leq 360^\circ$, determine the values of p, q and r.



2. Sketch the graph of $y = \tan x$ for $0^\circ \leq x \leq 360^\circ$ by setting up a table of values.

x	0	45	90	135	180	225	270	315	360
$\tan x$									

3.The diagram shows a graph of $y = \sin x$ for$0^\circ \leq x \ \leq 360^\circ$, determine the values of p, q and r.



4. Sketch the graph of $y = \cos x$ for $0^\circ \leq x \leq 360^\circ$ by setting up a table of values.

x	0	45	90	135	180	225	270	315	360
$\cos x$									

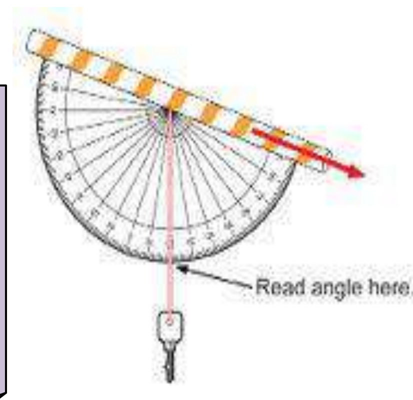
5. Sketch the graph of $y = \sin x$ for $0^\circ \leq x \leq 360^\circ$ by setting up a table of values.

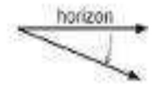
x	0	45	90	135	180	225	270	315	360
$\sin x$									

LEARNING OUTCOMES

Students should be able to:

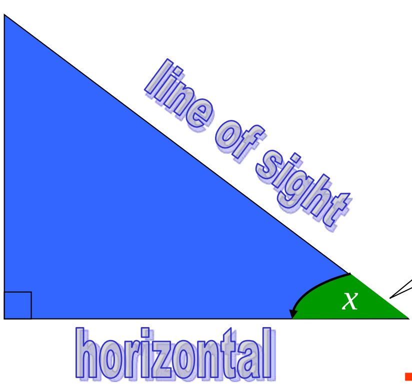
- Use a clinometer to measure angles in practical situations.
- Use angle measured above to calculate the height or length of object being studied





- A clinometer is an instrument for measuring angles of slope elevation or depression.

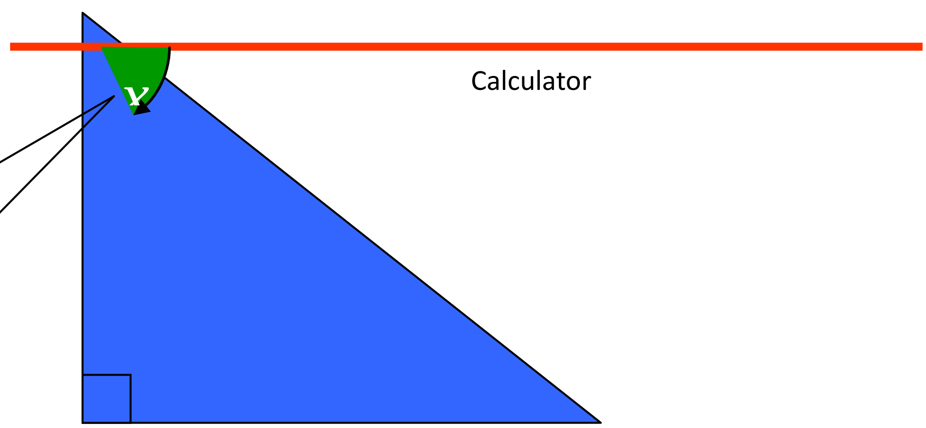
Angle of elevation



The angle of elevation is the angle formed by the line of sight and the horizontal

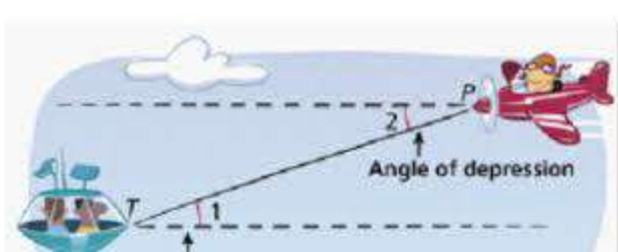
Angle of depression

The angle of depression is the angle formed by the line of sight and the horizontal

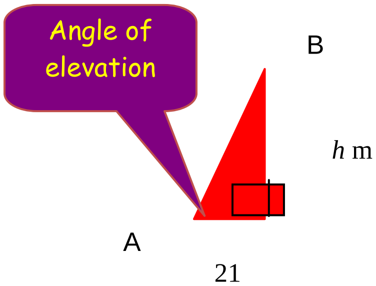


Example 4.15

The angle of elevation of building A to building B is 25^0. The distance between the buildings is 21 metres. Calculate how much taller Building B is than building A.



Step 1: Draw a right angled triangle with the given information.



Step 2: Take care with placement of the angle of elevation

Step 3: Decide which trig ratio to use.

$$\tan 25° = \frac{h}{21}$$

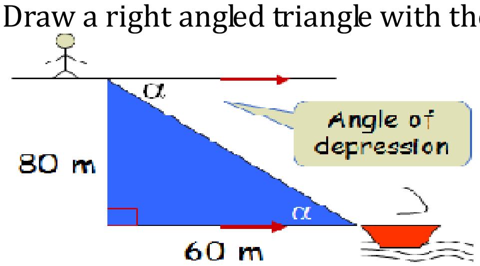
Step 4: Solve the trig equation.

$$h = 21 \times \tan 25°$$
$$h = 9.8\,m\ (1\,dec.\,pl)$$

Example 4.16

A boat is 60 metres out to sea. Madge is standing on a cliff 80 metres high. What is the angle of depression from the top of the cliff to the boat?

step One Draw a right angled triangle with the given



information.

Step two Identify the angle of depression

Step 3: Decide which trig ratio to use. $\tan\alpha = \frac{80}{60}$

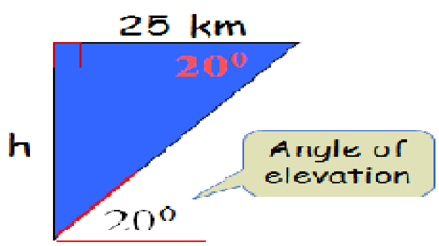
Step 4: Use calculator to find the value of the unknown. $\sqrt{3}\ \sqrt{11}\ \sqrt{8}$ $\alpha = 53.8'$

Exercise 4.7

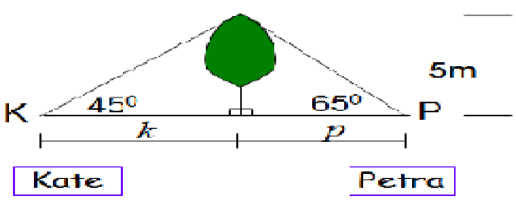
1. Neil sees a rocket at an angle of elevation of 11 degrees. If Neil is located 5 miles from the rocket's launchpad, how high is the rocket? Round your answer to the nearest hundredth.

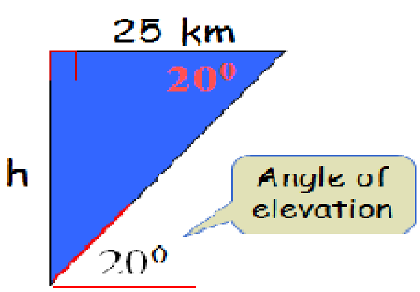
2. A boat is 500 meters from the base of a cliff. Jackie, who is sitting in the boat, notices that the angle of elevation to the top of the cliff is 32°. How high is the cliff? (Give your answer to the nearest metre).

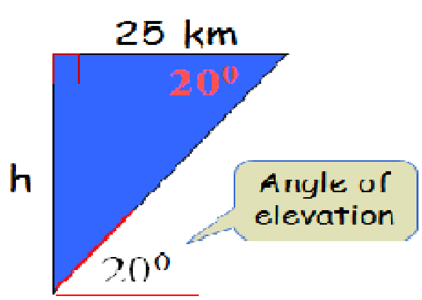
3. Marty is standing on level ground when he sees a plane directly overhead. The angle of elevation of the plane after it has travelled 25 km is 20^0. Calculate the altitude of the plane at this time

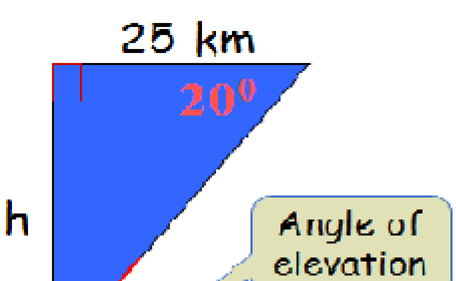


4. Kate and Petra are on opposite sides of a tree. The angle of elevation to the top of the tree from Kate is 45^o and from Petra is 65^o. If the tree is 5 m tall, who is closer to the tree, Kate or Petra





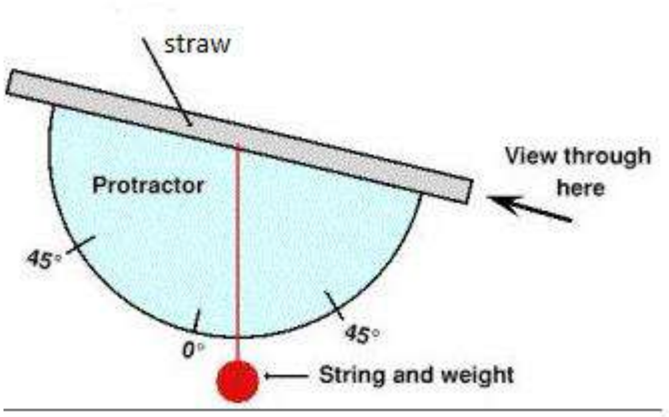




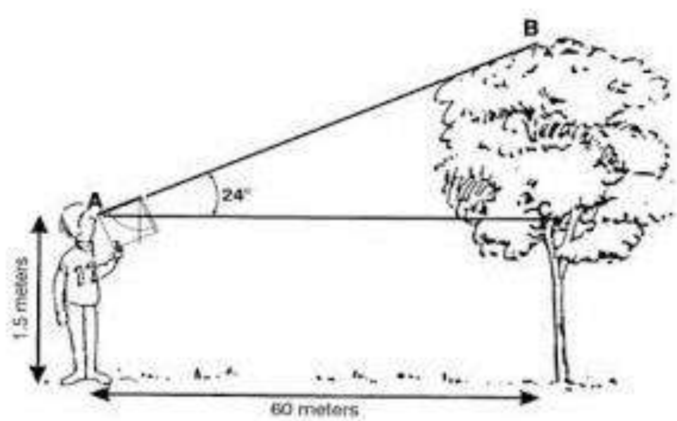
Let’s make your own clinometer. To make your clinometer you need the following:

1. A drinking straw 2. A protractor 3. Some adhesive 4. A small mass
   5. string

Use the adhesive to attach the straw to the protractor.



2. Let’s measure some inaccessible heights in your school.
   (a) You must choose your partner.
   (b) Obtain a measuring tape and measure each other’s height up to the eye level only.
   (c) Use the measuring tape to find an appropriate distance back from the object you are finding the height of.
   (d) Hold the clinometer level along the horizontal line and adjust the angle of the straw to sight the top of the object.
   Example



Exercise 4.8

Results

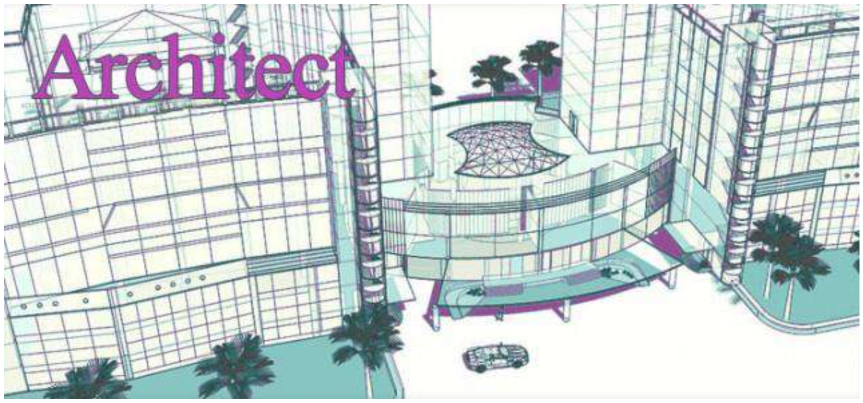
Object name	Height of person’s eyes From ground level	Angle of elevation	Distance from base	Height of object (show all working)

1. If you were to measure the height of a street light would you use a clinometer .Explain why or why not?
2. Jonetani is standing 15m from the base of a building and using a clinometer he measures the angle of elevation to be 37°. If his eyes are 1.65m above the ground, find the height of the building.

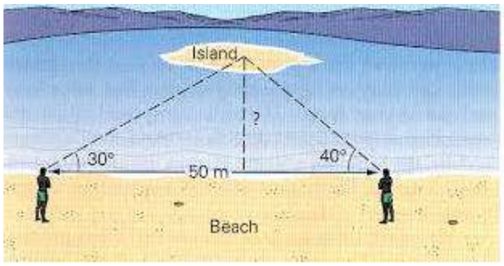
LEARNING OUTCOMES

Students should be able to:

- Identify some career opportunities related to trigonometry

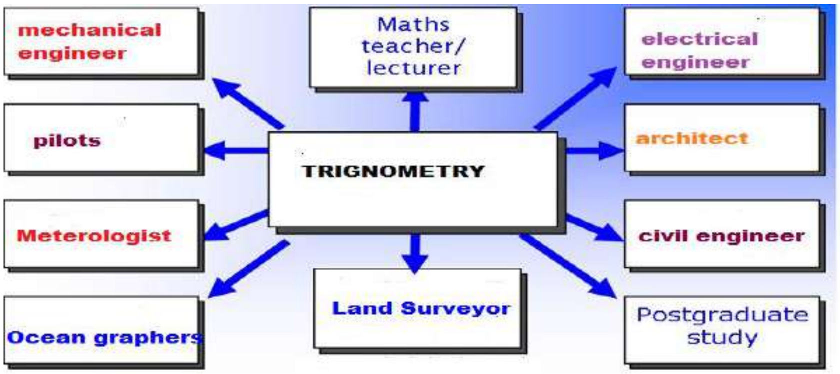


Trigonometry, out of all other topics is Mathematics is the most practical. It is almost used everywhere in our daily lives.

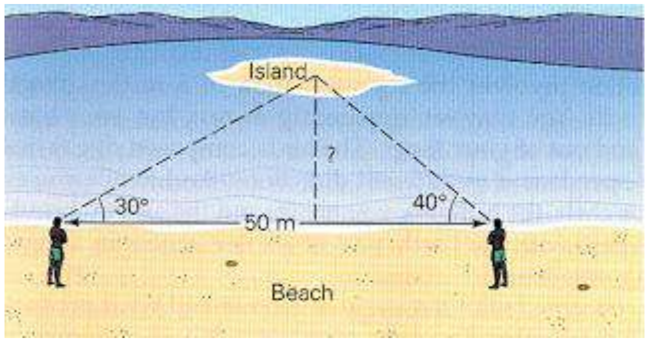


Applications of trigonometry include its study in physics, engineering and chemistry. Though trigonometry tables were created over two thousand years ago for computations in astronomy.

There are numerous jobs that require trigonometry of which some are shown below.



Trigonometry in real life situations



- Trigonometry is commonly used in finding the height of towers and mountains.

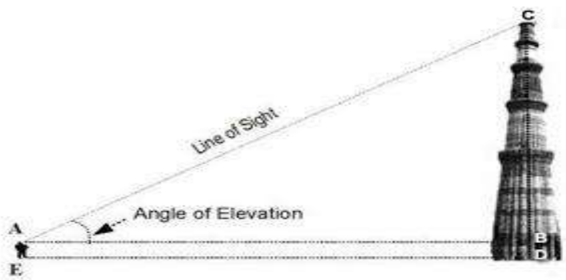
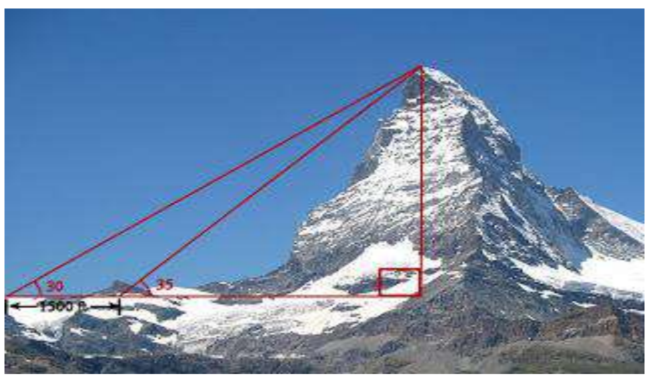
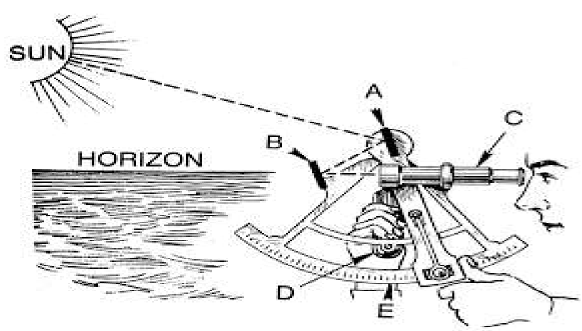


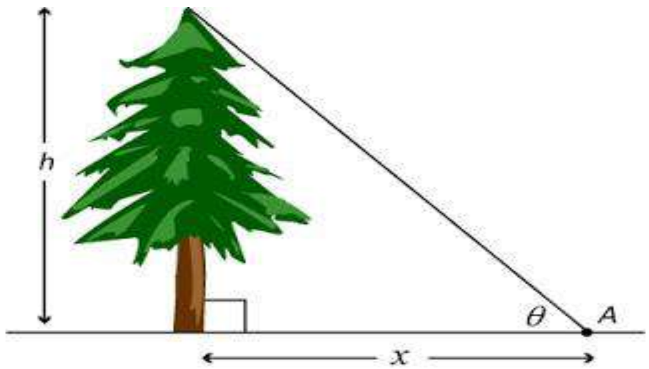
Fig 6



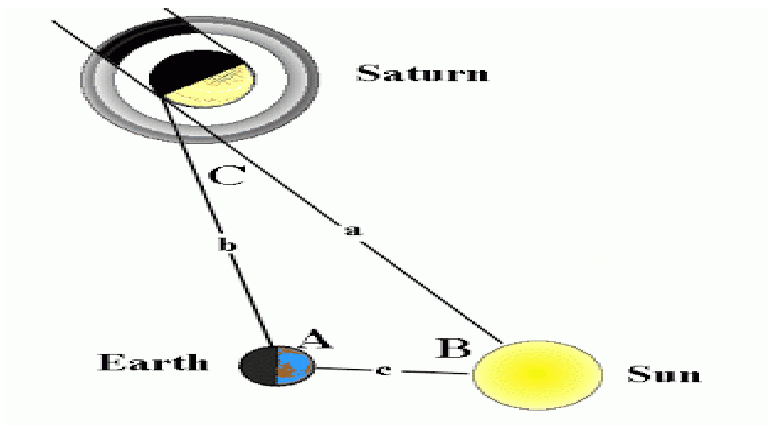
- It is used in navigation to find the distance of the shore from a point in the sea.



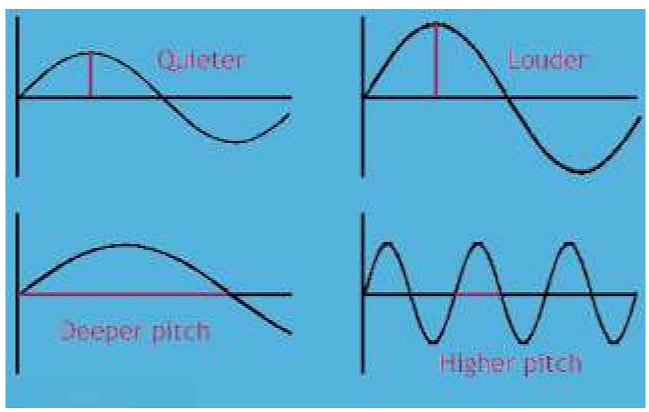
- It is used in oceanography in calculating the height of tides in oceans



- It is used in finding the distance between celestial bodies



- The sine and cosine functions are fundamental to the theory of periodic functions such as those that describe sound and light waves.



- Architects use trigonometry to calculate structural load, roof slopes, ground surfaces and many other aspects, including sun shading and light angles

Source: http://malini-math.blogspot.com/2011/08/applications-of-trigonometry-in-real.html

4.4 Constructions

4.4.1 Construct Angles

LEARNING OUTCOMES

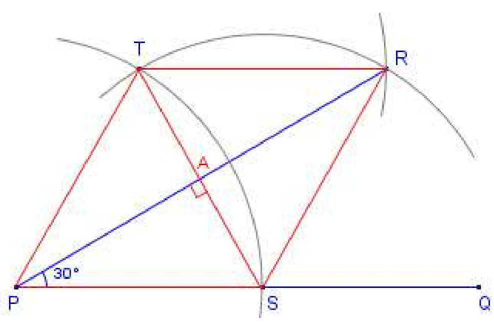
Students will be able to:

- Construct various special angles
- Construct angle bisector, midpoint and mediator of line segment
- Construct centers of triangles using ruler and compass

4.4.1.1 Constructing a 30° angle

Steps

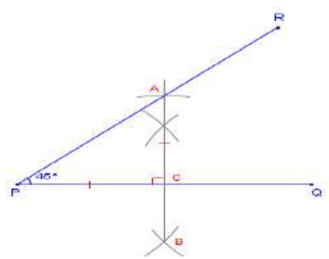
1. Draw a line PQ that will be one leg of the angle
2. With the compass point on P, set it on any width between P and Q
3. Draw an arc crossing PQ at S
4. Move the compass point to S and draw another arc crossing the first one at T
5. Move to T and make an arc crossing the previous one , labelling the intersection point R
6. Draw a straight line from P through R
7. Angle RPQ = 30^O



4.4.1.2 Constructing a 45° angle

Steps

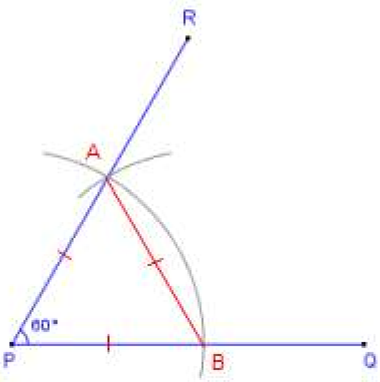
1. Draw a line PQ that will be one leg of the angle
2. With the compass point on P, set it on any width of more than half the length of PQ
3. From points P and Q, draw arcs above and below line PQ
4. Draw a straight line joinning the arc intersections. The intersection of the two straight lines is the midpoint of line PQ.
5. From the midpoint of PQ, set the compass width to point P
6. Draw an arc across the perpendicular line and label it Point C
7. Draw line PC
8. Angle PCQ = 45^{O}



4.4.1.3 Constructing a 60° angle

Steps

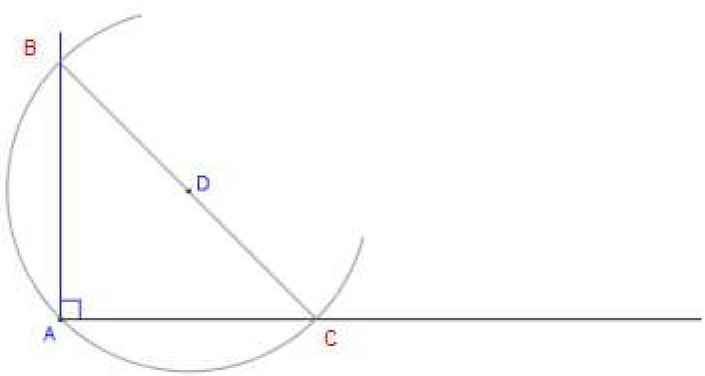
1. Draw a line PQ that will be one leg of the angle
2. With the compass point on P, set its width to about half of PQ
3. Draw an arc from above point P and crossing PQ
4. Place the compass point to where the arc crosses PQ, draw an arc above PQ and ensure that the two arcs cross each other. Label the point of intersection R
5. Draw straight line PR
6. Angle RPQ = 60^{O}



4.4.1.4 Constructing a 90^o angle

1. Draw a horizontal straight line and label one end A
2. Mark point D somewhere above and between the end points of the line drawn
3. Place the compass point on D and sets its width to point A
4. Draw an arc across the line to above point A
5. Draw a diameter through D starting from where the arc crosses the line
6. Draw a stright line from A to the other end of the diameter

7 The angle between the two straight lines is 90^O

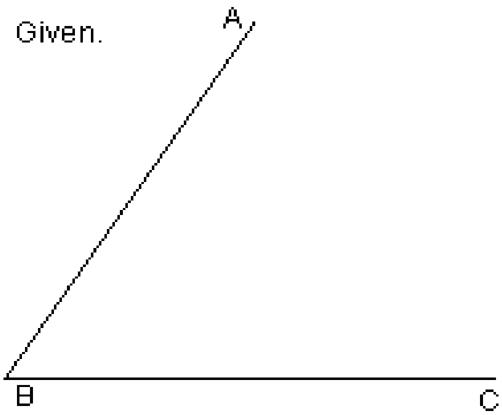


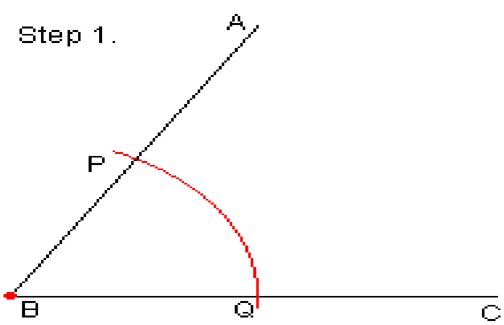
4.4.1.5 Angle Bisector

Steps

Step 1

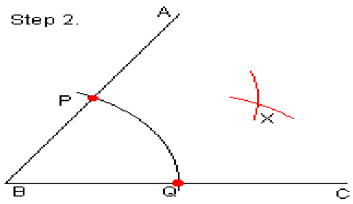
With the compass point at the centre of the vertex of the angle, draw an arc with radius of any length. The arc must be intersecting both sides of the angle. Label the intersection points P and Q





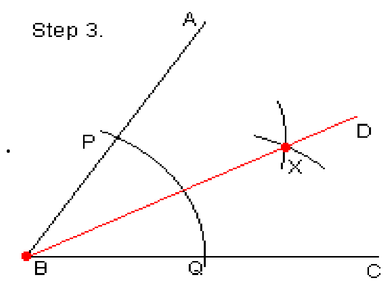
Step 2

Place the compass point at point P , draw an arc of any length within lines AB nad BC. Place the compass at point Q and do the same. The radius of the second arc must be the same as the first and the two arcs must be long enough to intersect at a point X. .



Step 3

Draw a straight line from the center of the vertex through the point of intersection X.



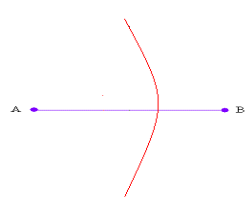
4.4.1.6 Constructing the midpoint of a line segment

The midpoint of a line segmentis its middle point which is equidistant from the end points.

Constructing the midpoint of a segment is considered to be quite simple and easier than any other straightedge construction.

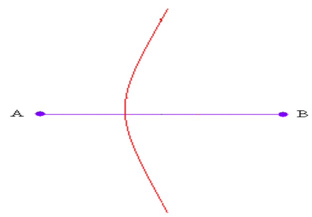
Step 1

Draw line segment AB. Place the compass point at point A and with the width of more than half the length of AB draw an arc from a point above AB right down to a point below AB

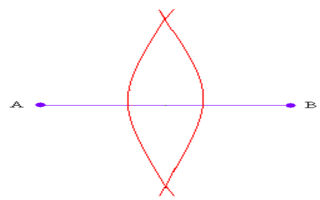


Step 2

Now place the compass point at B and draw an arc

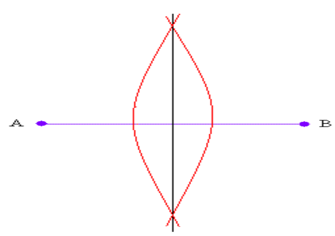


The two arcs should intersect above and below AB

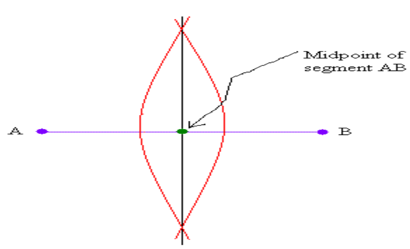


Step 3

Draw a straight line joining the two point intersection

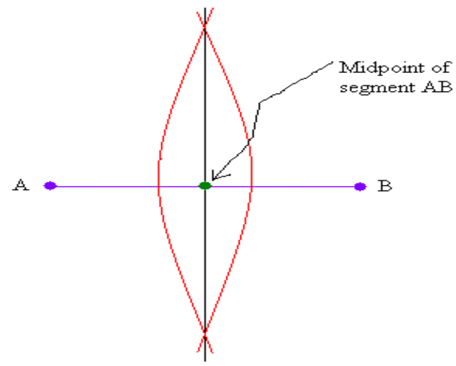


The point of intersection of the vertical line and the line segment AB shown with a green point is the midpoint



4.4.1.7 Constructing the Mediator of a line segment

A **mediator** is the same as a **perpendicular bisector**, The bisector of segement AB is the line perpendicular to this segment which passes through its midpoint

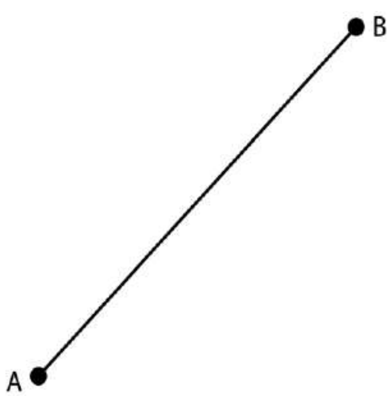


Exercise 4.8

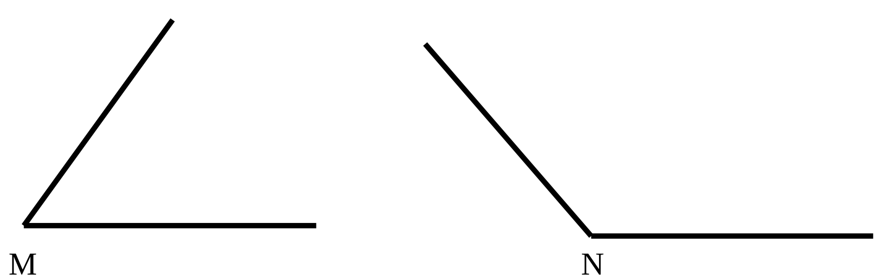
1. Bisect a 30^o angle

2. Bisect a 45^o angle

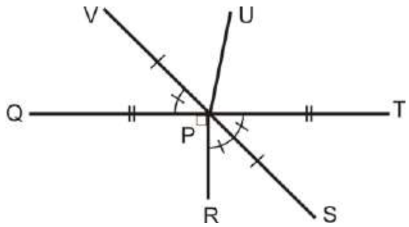
3. Bisect the line segment AB given below



4. Construct the bisector of $\angle M$ and $\angle N$



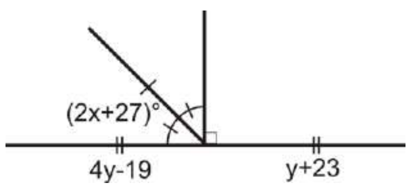
5. Use the following picture to answer the questions.



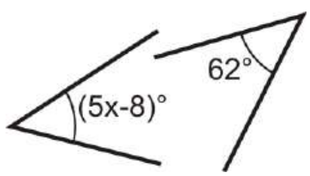
(a) What is the angle bisector of ∠TPS?
(b) P is the midpoint of what two segments?
(c) Find ∠QPR
(d) Find ∠TPS

6. Use algebra to determine the value of variable(s) in each problem.

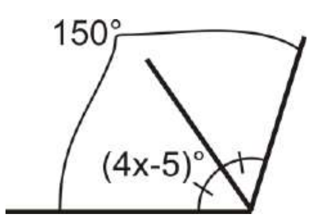
(a)



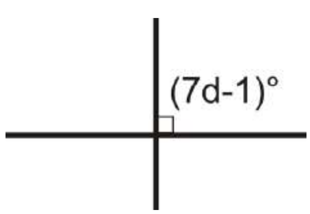
(b)



(c)



(d)

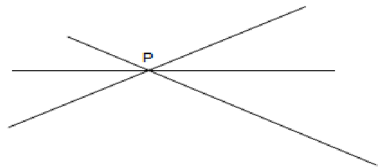


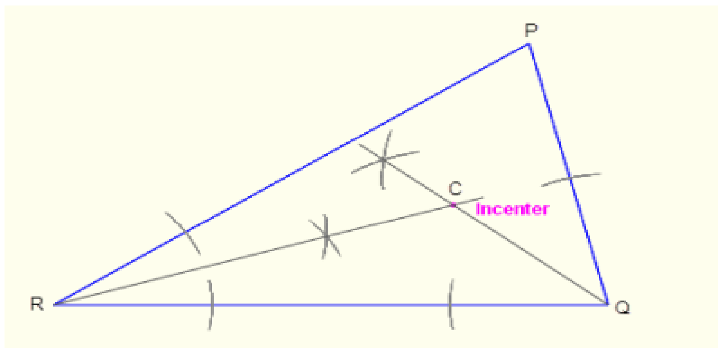
Constructing centers of triangles

4.4.2 The Incenter

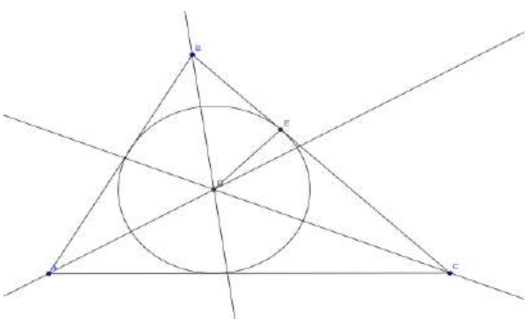
Definition: The point of concurrency of the **angle bisectors** is called the **incenter** of the triangle.

Points of currency: The point where three or more lines intersect and usually refers to various centers of a triangle.





The circle that has its center at the incenter and is tangent to each of the sides of the triangle is called the *inscribed circle*, or simply the *incircle* of the triangle.



The steps for constructing the incenter and incircle are provided below

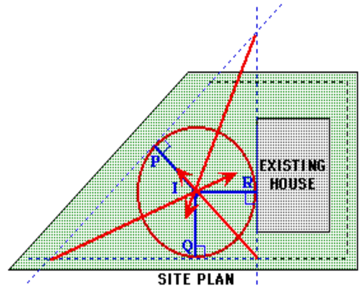
1. Draw a triangle
2. Construct the angle bisectors of each of the interior angles. These bisectors will intersect in a single point called the incenter.
3. Mark the point of intersection of these lines.

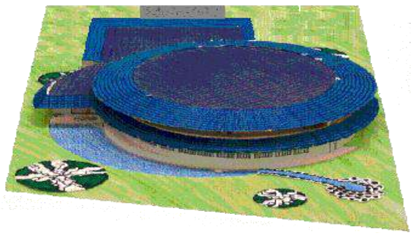
4. Draw a perpendicular line from the incenter to one of the sides of the triangle. This line becomes a radius.
5. Construct a circle that has center of the incenter and the intersection point on the circle itself. This is the incircle.

Real life examples

Question : You are an architect and are given a task to design a round office tower. The client wants the tower to be circular. The location on which it will be built is at the corner of two streets and there is another buildng located on the property Where would you place the center of the lanai (A veranda or roofed patio), to make it as big as possible?

Answer: The center of the round house needs to be equidistant from the street and the two properties lines: if we must therefore be equidistant from the 3 sides of the triangle formed by the 3 lines. If we consider 2 lines at a time, the set of points equidistant from two lines is the bisector of the angle between 2 lines. So the set of points equidistant from all 3 lines is the point where all 3 angle bisectors meet. This point is called the Incenter of the triangle. Because it is equidistant from the 3 sides of the triangle, if we use that distance as radius, we can construct a circle tangent to all 3 sides of the triangle. This circle is the outline of our lovely new round house





Source: https://www.google.com/search?q=google&ie=utf-8&oe=utf-8#q=incenter+of+a+triangle+in+real+life

- The **incenter** could be used to build a clock. You wouldn't want the hands on the clock to be off centered so you would find the middle of the circle.

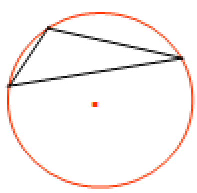
- Three pilots are flying over a triangular city. They plan to attack. The approach the city at the corners, or verticies. They bisect each angle. They all are traveling, and have a 3 plane collision in the center of the city. Too bad they hadn't calculated the incenter.

Source: http://www.illustrationsof.com/royalty-free-airplane-clipart-illustration-32931.jpg

4.4.3 The Circumcenter

Circumcenter is the center of a triangle's circumcircle. It is where the "perpendicular bisectors" (the lines perpendicular to the side of each triangle which passes through its midpoint) meet.

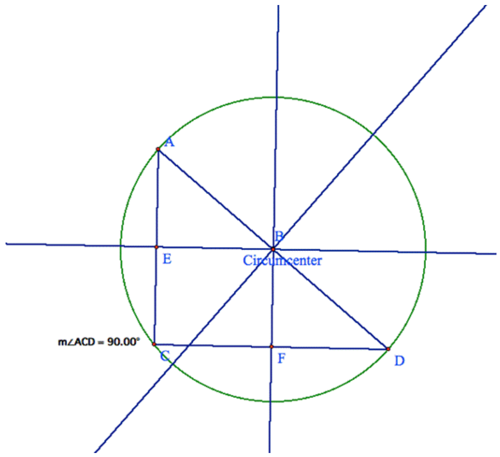
The circumcircle (circumscribed circle) of a polygon is a circle which passes through all the vertices of the polygon. The center of this circle is called the circumcenter.



In the case of a triangle, there is always a circumcircle possible, no matter what shape the triangle is.

The circumcenter is constructed by identifying the midpoints of the segments AC, CD, and DA. Then a perpendicular line is drawn through the midpoints perpendicular to the side segment.

The example shown below is of a right – angled triangle.



Note that the triangle is inside the circle and the circumcenter is in the center of the circle.

Constructing the Circumcenter

Use these steps to construct the circumcenter of the given triangle (ΔACD).

1 Construct the perpendicular bisector of one side (CD).

2. Construct the perpendicular bisector two other sides: AC and DA.
3. Where the 3 perpendicular bisectors intersect is the circumcenter. Mark this as point B.
4. Place the compass point on B. Open the compass so that the pencil is on any vertex. of the triangle. Draw a circle with this compass width with the center at B.

Real Life Examples

There are many reasons people may need to know the altitude of buildings. Pilots need to know the altitude of a building if they are going to be landing on a building, such as the pilots that fly the flight for life helicopter.

Source: https://www.google.com/search?q=google&ie=utf-8&oe=utf-8#q=images+of+helicopters

Without knowing the altitude of the building the pilot wouldn't know how fast or slow to begin going and when to position the jet to get it on the building. Altitdue plays a very important role in the career of Piloting.

There are many uses of the circumcenter of a triangle in real life, but most of these revolve around one main thing. That is when you are locating a main point which is the same distance to three different locations or spots making it most relevant to people. First of all circumcenter is where three perpendicular bisectors meet at the point of concurrency meaning that spot would be equal to all three vertices.

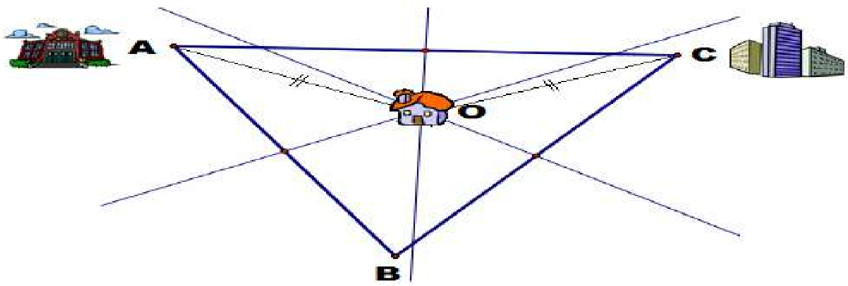
- A company is designing a mall. They want the food court to be in the very center, and equidistant from the three main stores (or verticies). They would put it at the circumcenter.

Source:
http://www.picturesof.net/_images_300/People_At_A_Shopping_Mall_Royalty_Free_Clipart_Picture_081106-160290-408048.jpg

- Finding the **circumcenter** could also be used when building a house. If you wanted to put a window in the middle of a wall then you could find the circumcenter to do that.

Source: https://www.google.com/search?q=google&ie=utf-8&oe=utf-8#q=circumcenter+of+a+triangle+in+real+life

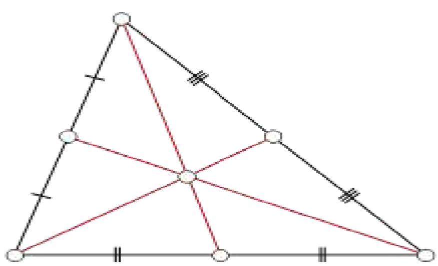
- a family wanted to buy a home, but they wanted it to be close both to both the children's school and the parents' workplace. By looking at a map, they could find a point that is equidistant from both the workplace and the school by finding the ***circumcenter*** of the triangular region.



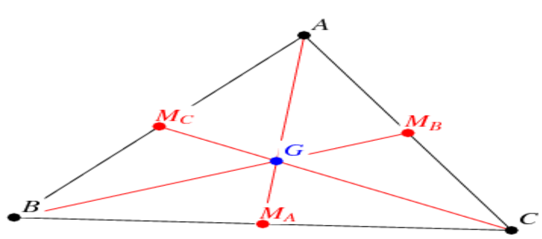
4.4.4 The Centroid

A **centroid of a triangle** is the point where the three **medians** of the triangle intersect.

A **median** of a triangle is a line segment joining the vertex to the mid - point on the opposite side of the triangle.



The **centroid** is also referred to as the center of gravity of the triangle



Centroid facts

- The centroid is always inside the triangle
- Each median divides the triangle into two smaller triangles of equal area.
- The centroid is exactly two-thirds the way along each median.
  Put another way, the centroid divides each median into two segments whose lengths are in the ratio 2:1, with the longest one nearest the vertex. These lengths are shown on the one of the medians in the figure at the top of the page so you can verify this property for yourself.

Constructing the Centroid

The steps for constructing centroid are given below:

1. Draw Triangle *ABC*

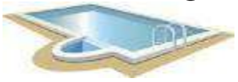
2. Construct the mid - point of each side of the triangle and label them $M_{A,}$ M_B, M_C accordingly

3. Draw a line joining the mid – point of *BC* to the vertex *A* on the opposite side (median). Repeat this for the other two sides and vertices

4. Label the point of intersection of these three medians *G* which is called the **Centroid**

Source: https://www.google.com/search?q=google&ie=utf-8&oe=utf-8#q=centroid+in+real+life&start=10

The **centroid** of a triangle could be used in real life by needing to find the center of a certain area. For example someone is putting a swimming pool in the center of a community they will need to find right where the middle is.



A Centroid; A magician is preforming a show, and he needs an object to remain on a triangular stand as he pulls away a cloth from beneath. The centroid is the center of gravity. He would be successful if the object was placed on the centroid.

Other real life situations

Baseball pitch

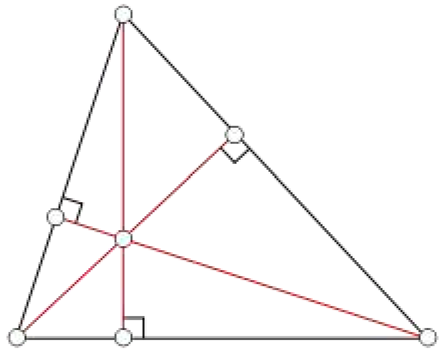
Source: https://www.google.com/search?q=google&ie=utf-8&oe=utf-8#q=images+of+baseball+fields

Roundabout

Source: https://www.google.com/search?q=google&ie=utf-8&oe=utf-8#q=images+of+roundabouts

4.4.5 The Orthocenter

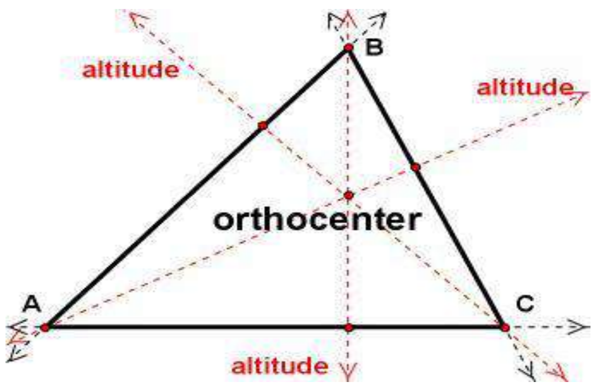
The **orthocenter** of a triangle is the intersection of the three **altitudes** of a triangle. The **altitude** of a triangle is a **perpendicular** segment from the vertex of the triangle to the opposite side.



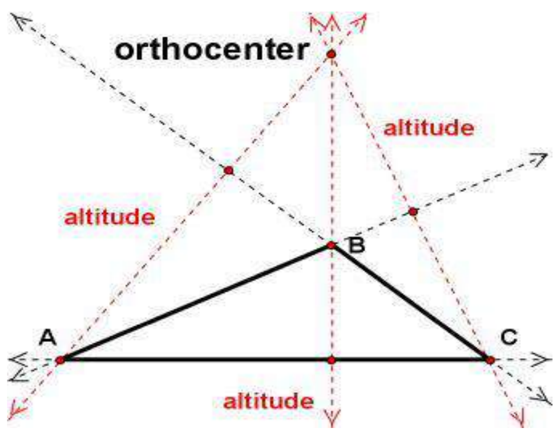
There are therefore three altitudes possible, one from each vertex.

The orthocenter is not always inside the triangle. If the triangle is obtuse, it will be outside. To make this happen the altitude lines have to be extended so they intersect.

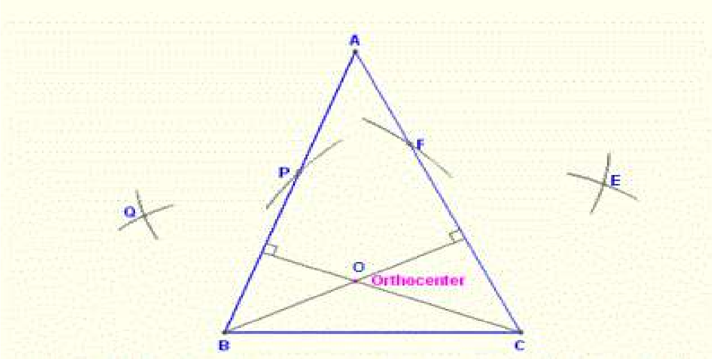
Orthocenter of an Acute triangle



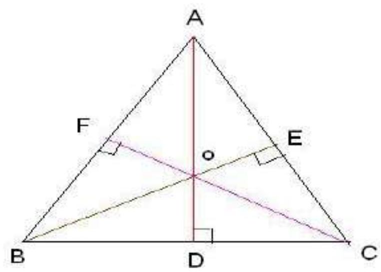
Orthocenter of an Obtuse triangle



Constructing orthocenter



If the three coordinates A, B and C are given, the following procedure is to be followed to find the orthocenter:



Steps

1. Draw triangle ABC
2. Select vertex A and segment BC
3. Construct an altitude from vertex A to segment BC
4. Repeat this for the other two vertices and segments
5. The point where these altitudes intersect is the orthocenter

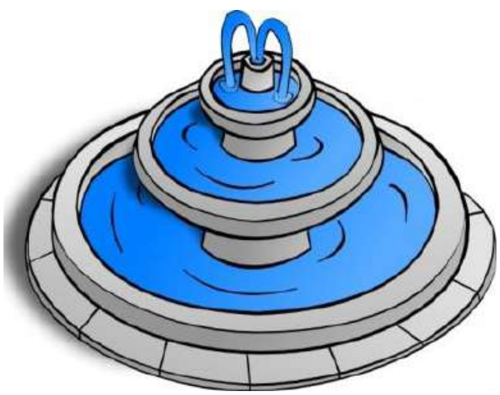
Real Life Examples

Orthocenter:
Imagine that you still live at a vertex of Denny Triangle. You want to find the shortest distance you must walk to get to the street that is the opposite side of the triangle. Since a straight line is the shortest distance, finding the street that is perpendicular to the opposite Would give you the shortest distance. Finding the orthocenter would give the perpendicular line, or altitude, from any vertex.

- Another example of **orthocenter** is the **Eiffel tower**. They might of use the orthocenter to find where all the altitudes met while building it.

- In a triangular format lay a mall, park, and hotel. The city wants to build a memorial fountain visible from every site. They find the altitudes, then the orthocenter for everyone to enjoy.

Source: http://images.all-free-download.com/images/graphiclarge/fountain_clip_art_17015.jpg



Point of concurrency

The Centroid is the point of concurrency where the 3 medians of a triangle meet. This point is also the triangle's center of gravity.

2) The Circumcenter is the point of concurrency where the perpendicular bisectors of all three sides of the triangle meet. This point is the center of the triangle's circumscribed circle.

3) The Incenter is the point of concurrency where the angle bisectors of all three angles of the triangle meet. Like the circumcenter, the incenter is the center of the inscribed circle of a triangle.

4) The Orthocenter is the point of concurrency where the 3 altitudes of a triangle meet. Unlike

the other three points of concurrency, the orthocenter is only there to show that altitudes are concurrent. Thus, bringing me back to the initial statement.

Exercise 4.9

1. Write True or False

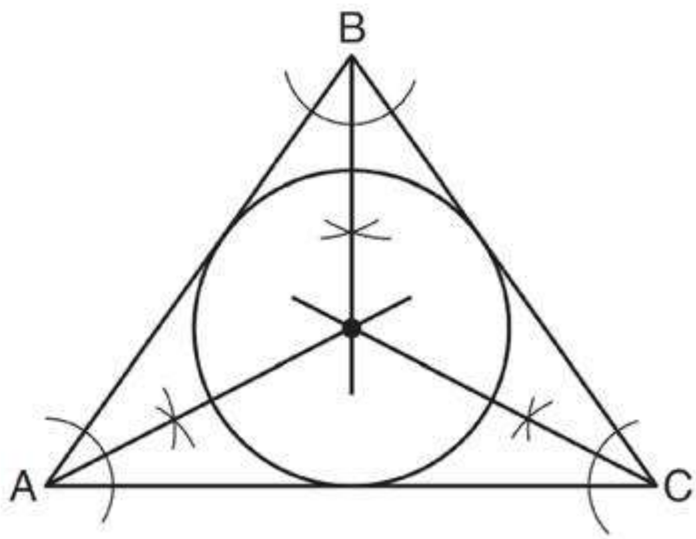
   (a) The angle bisectors of a scalene triangle intersect outside the triangle ___________

   (b) The altitude from the vertex angle of an isosceles triangle is always the median__

   (c) To find the point that is equidistant from the sides, find the circumcenter ______

   (d) The median starts at a vertex and goes to the opposite midpoint ______________

2. Which principle is used in the construction shown below?



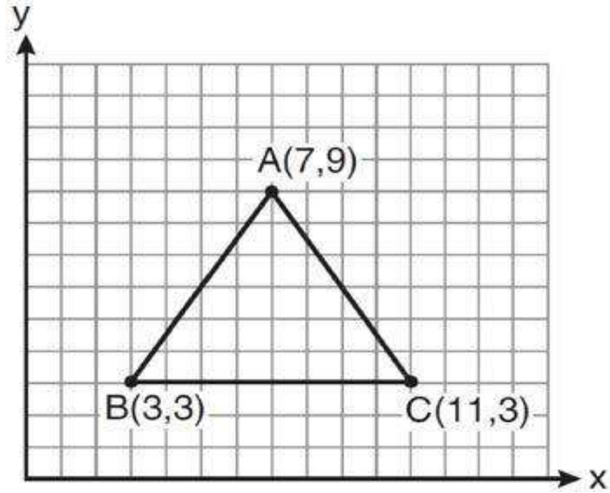
   A. The intersection of the angle bisectors of a triangle is the center of the inscribed circle

   B. The intersection of the angle bisectors of a triangle is the center of the circumscribed circle.

   C. The intersection of the perpendicular bisectors of the sides of a triangle is the center of the inscribed circle.

   D. The intersection of the perpendicular bisectors of the sides of a triangle is the center of the circumscribed circle.

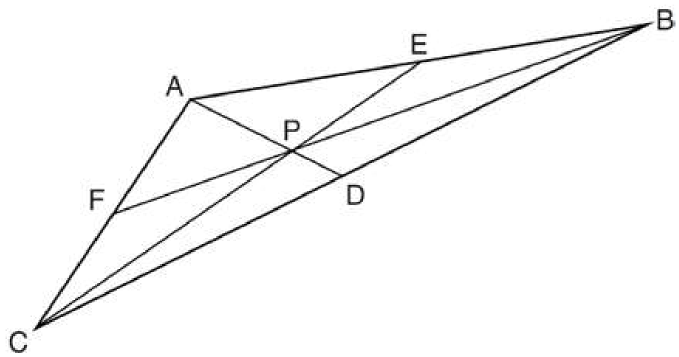
3. The vertices of the triangle drawn are $A(7, 9)$, $B(3, 3)$, and $C(11, 3)$.



What are the coordinates of the centroid of △ABC?

A. (5, 6)
B. (7, 3)
C (7, 5)
D (9, 6)

4 In the diagram below of ABC, AE ≅ BE, AF ≅ CF, and CD ≅ BD.



The point marked P is known as the

A Incenter
B Circumcenter
C Orthocenter
D. Centroid

5 The coordinates of the endpoints of *AB* are *A*(0, 0) and *B*(0, 6). The equation of the perpendicular bisector of *AB* is

A. $x = 0$ B. x = 3

C. y = 0 D. y = 3

4.5 Intersecting Chord Theorem

Learning outcomes

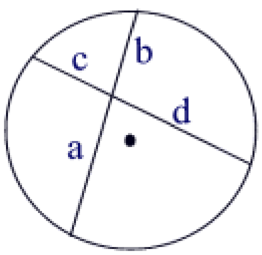
Students will be able to:

- Identify different types and properties of intersecting chords.
- Prove properties of intersecting chords
- Apply intersecting chords to real life

4.5.1 Two Chords Intersect in a Circle

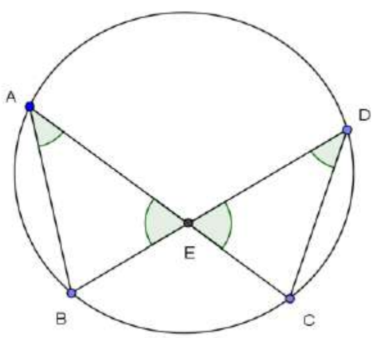
Theorem 1

If two chords intersect inside a circle then the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord



$a \bullet b = c \bullet d$

Theorem Proof:



Given chords AB and CD, prove that $\overline{AE} \bullet \overline{EC} = \overline{BE} \bullet \overline{ED}$

$\angle BAC = \angle BDC$ as inscribed angles subtended by the same chord BC

∠ABD = ∠ACD   as inscribed angles subtended by the same chord AD

∠AEB = ∠DEC   as a pair of vertical angles

By Triangles with Two Equal Angles are Similar we have $\triangle AEB \sim \triangle DEC$.

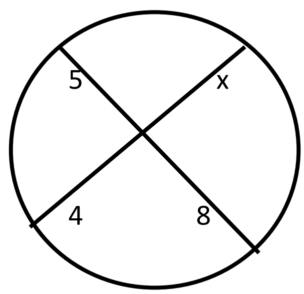
Thus:

$$\frac{AE}{EB} = \frac{DE}{EC}$$

$$\Longrightarrow AE \cdot EC = DE \cdot EB$$

Example 4.17 Find x in each of the following

(a)

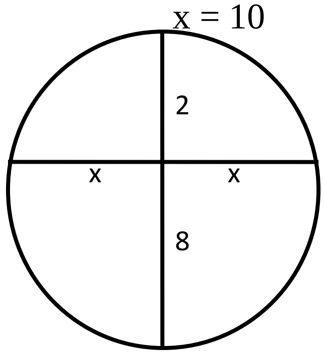


By Theorem 1, $4 \bullet x = 5 \bullet 8$

$4x = 40$

$x = 10$

(b)



By Theorem 1, $x \bullet x = 2 \bullet 10$

$x^2 = 20$

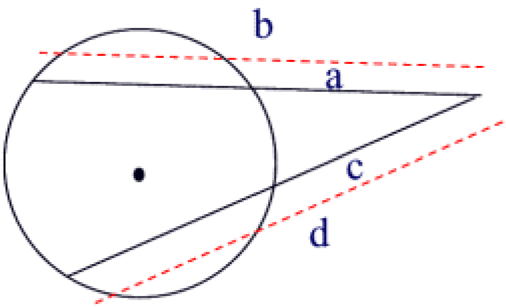
$x = \sqrt{20}$

$x = 4.47$

4.5.2 Two Secant Segments

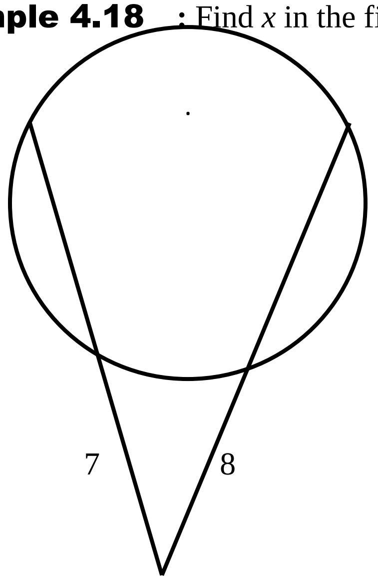
Theorem 2

If two secant segments are drawn to a circle from the same external point, the product of the length of one secant segment and its external part is equal to the product of the length of the other secant segment and its external part.



$a \cdot b = c \cdot d$

Example 4.18 : Find *x* in the figure below



By Theorem 2, $\quad \mathbf{(x + 7) \cdot 7 = (6 + 8) \cdot 8}$

$$\mathbf{7x} + 49 = 14 \cdot 8$$

$$7x = 112$$

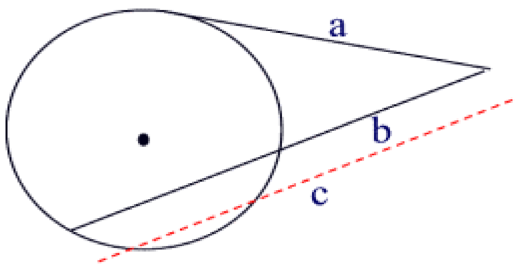
$$x = 112/ 7$$

$$x = 16$$

4.5.3 A Tangent and a Secant Segment

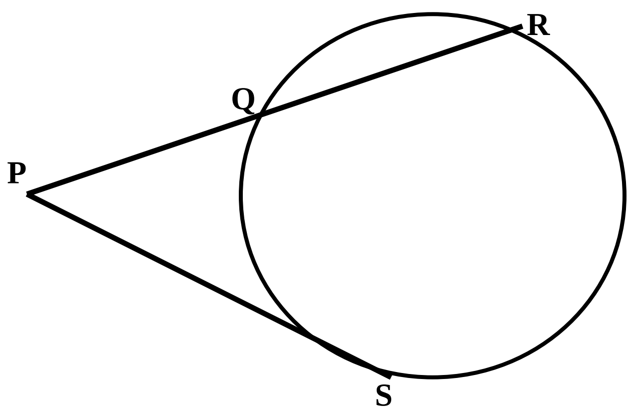
Theorem 3

If a secant segment and tangent segment are drawn to a circle from the same external point, the product of the length of the secant segment and its external part equals the square of the length of the tangent segment.



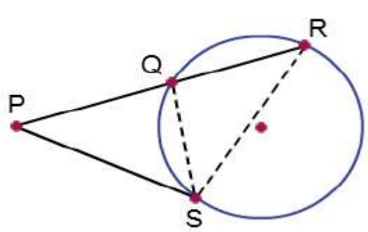
$b \cdot c = a^2$

Given secant PR and tangent PS, prove that $PS^2 = PR \times PQ$



Theorem Proof:

In order to prove this theorem, we are going to join QS and RS.



In ΔPSQ and ΔPRS

$\angle PSQ = \angle PRS$ ______ (By tangent-chord theorem, according to which an angle formed by a tangent and a chord is equal to the angle formed by that chord at another point at the circle.)

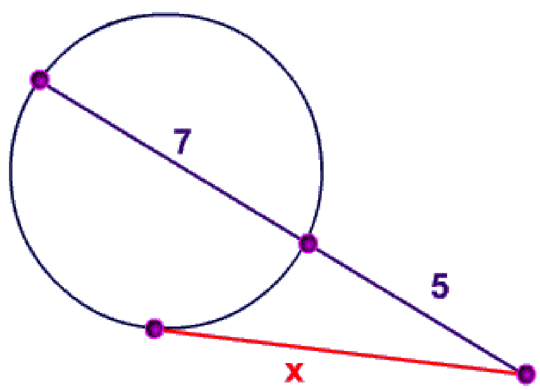
angle QPS = angle RPS ______ (Common angle)

Thus, $\triangle PSQ \sim \triangle PRS$

$$\frac{PS}{PR} = \frac{PQ}{PS}$$

$PS^2 = PR \times PQ$ **Hence proved.**

Example

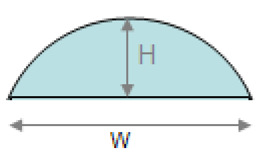


Use the theorem for the intersection of a tangent and a secant of a circle to solve the problems below.
In the diagram on the left, the red line is a tangent, how long is it?

$x^2 = (7+5) \cdot 5$
$x^2 = (12) \cdot 5$
$x^2 = 60$
$x = \sqrt{60}$

$x = 7.75$

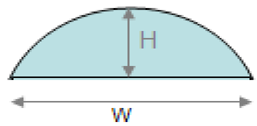
A Practical use



When making doors or windows with curved tops we need to find the radius of the arch so we can lay them out with compasses. See Radius of an Arc for a way to do this using the Intersecting Chords Theorem.

Circular arcs turn up frequently in the real world, such as the top of the window shown on the right. When constructing them, we frequently know the width and height of the arc and need to know the radius. This allows us to lay out the arc using a large compass to calculate the rad

Given an arc or segment with known width and height:



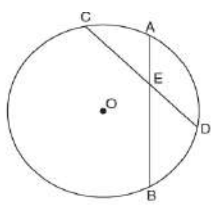
The formula for the radius is:

$$Radius = \frac{H}{2} + \frac{W^2}{8H}$$

where:
W is the length of the chord defining the base of the arc
H is the height measured at the midpoint of the arc's base.

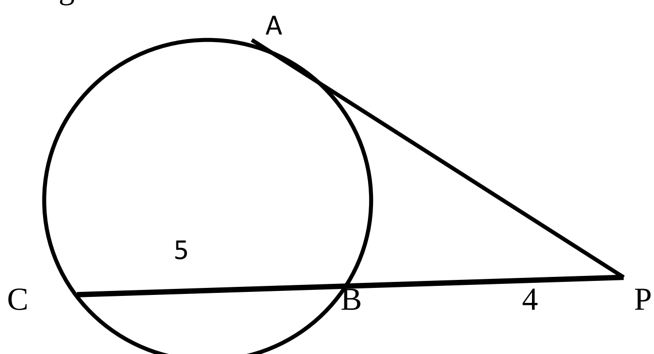
Exercise 4.10

1. In the circle given below, $\overline{AB}$ and $\overline{CD}$ intersect at E. If $\overline{CE}$ = 10 cm, $\overline{DE}$ = 6 cm and $\overline{AE}$ = 4 cm, what is the length of $\overline{EB}$?



A. 15cm
B. 12cm
C. 6.7cm
D 2.4cm

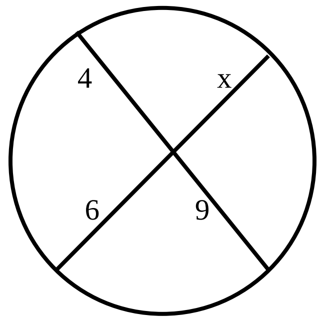
2. In the diagram below, tangent $\overline{PA}$ and secant $\overline{PBC}$ are drawn from external point P. What is the length of $\overline{PA}$?



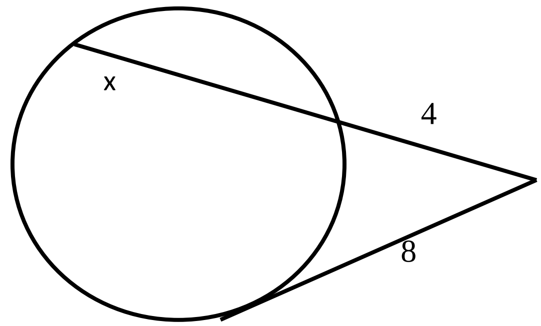
A. 20
B. 9
C. 8
D. 6

3. For each of the following, find the value of x

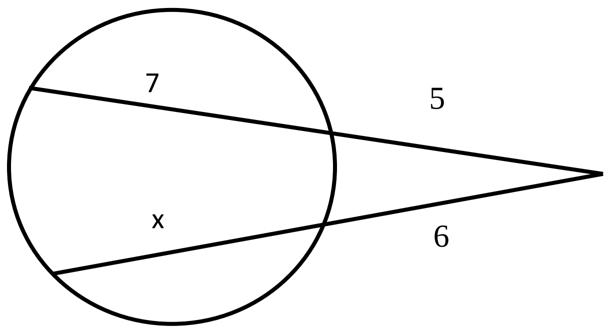
a



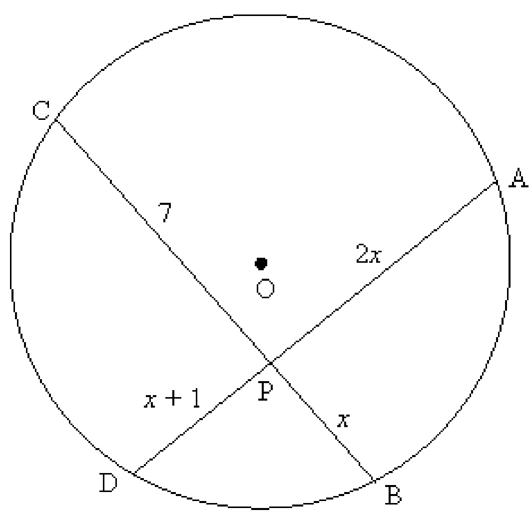
b.



c.



4. In the circle given below, chords CB and AD intersect at point P. The segments formed by these intersecting chords are CP = 7, BP = x, AP = 2x, and DP = x + 1. What is the measure of chord CB?



GLOSSARY		
1	Acute triangle	All angles are less than right angles
2	Adjacent side	Adjacent sides are those the are next to each other
3	Altitude	A line segment through a vertex and perpendicular to (i.e. forming a right angle with) a line containing the base (the opposite side of the triangle)
4	Angle bisector	A line segment that bisects one of the vertex angles of a triangle.
5	Angle of depression	Angle below the horizontal
6	Angle of elevation	Angle above the horizontal
7	Arc	A portion of the circumference of a circle
8	Bisect	To divide into two equal parts
9	Centroid	It is a point of concurrency of the triangle. It is the point where all 3 medians intersect and is often described as the triangle's center of gravity
10	Chord	It is a geometric line segment whose endpoints both lie on the circle
11	Circumcenter	The point where the three perpendicular bisectors of the sides of a triangle meet. Also, the center of the circumcircle. One of a triangle's points of concurrency.
12	Circumscribed circle	It is a circle which passes through all the vertices of the polygon. The center of this circle is called the circumcenter and its radius is called the circumradius
13	Clinometer	An instrument used in surveying for measuring an angle of inclination
14	Point of Concurrency	The point of intersection of the lines, rays, or segments
15	Construction	The act of drawing geometric shapes using only a compass and straightedge
16	Equidistant	When a point is the same distance from each figure.
17	Geometry	The branch of mathematics that treats the properties, measurement, and relations of points, lines, angles, surfaces, and solids.
18	Hypotenuse	The side of a right triangle opposite the right angle.
19	Incenter	The point of concurrency of the three angle bisectors of a triangle
21	Inscribed circle (incircle)	When a circle drawn in a triangle touches the sides
23	Median	A segment from a vertex to the midpoint of the opposite side
24	Mediator	The plane through the midpoint of a line segment and perpendicular to that segment
25	Midpoint	A point on a line segment that divides it into two equal parts
27	Orthocenter	It is where altitudes meet.
28	Perpendicular bisector	A segment, ray, line, or plane that is perpendicular to a segment at its midpoint
30	Pythagoras Theorem	The square of the length of the hypotenuse of a right triangle equals the sum of the squares of the lengths of the other two sides
31	Secant segment	A secant is a line that intersects a circle at two points.
32	Segment	The region bounded by a chord and the arc subtended by the chord
34	Straightedge construction	It is the construction of lengths, angles, and other geometric figures using only a ruler and compass.
35	Subtend	To be opposite to and delimit (an angle or side)
36	Tangent	A straight line to a plane or curve at a given point that "just touches" the curve at that point
38	Trigonometry	The branch of mathematics dealing with the relations of the sides and angles of triangles and with the relevant functions of any angles
39	Vertex	A vertex is a point where two or more straight lines meet. It is a Corner.