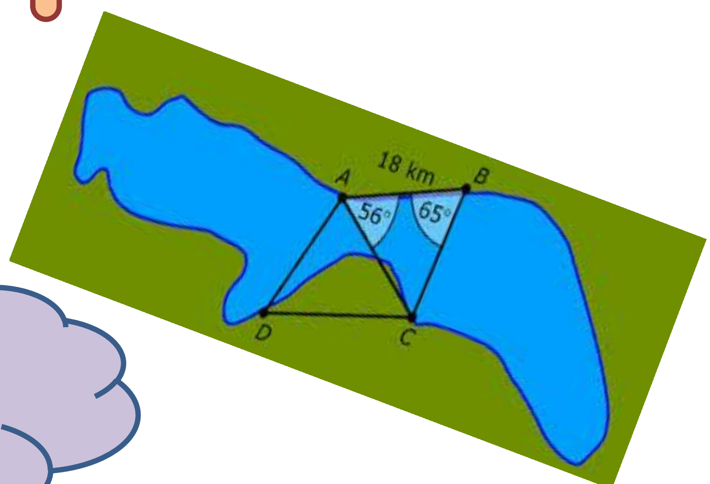
STRAND FIVE

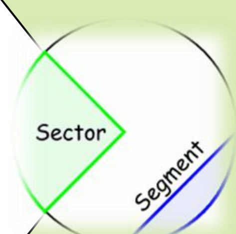
TRIGONOMETRY

SUB – STRAND 5.1

Non – Right Angle Triangles



CONTENT LEARNING OUTCOMES

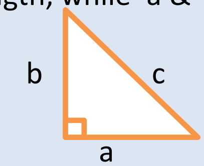


- Pythagoras Theorem and SOH/CAH/TOA
- Sine and Cosine Rule
- Exact values
- Area of Triangle
- Conversion of Angles
- Arc Length, Area of Sector and Segments
- Solving Trigonometric Equations

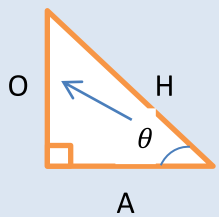
SINE and COSINE RULE

Note: Recall for **right – angle triangles**:

- Use Pythagoras theorem, i.e. $c^2 = a^2 + b^2$
  to solve for the unknown sides where c is the hypotenuse or the longest length, while ‘a’& ‘b’ are the other 2 lengths of the triangle.



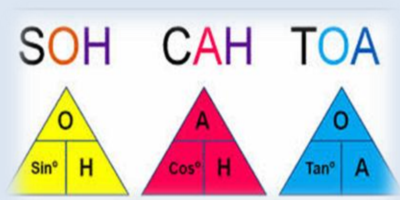
- Use SOH/CAH/TOA to solve for the unknown sides and angles, where O is opposite side of an angle, A is the adjacent side (i.e. next to angle) and H is the hypotenuse or longest side (i.e. opposite of right angle, 90^0).



$$\text{Sin } \theta = \frac{opposite}{hypotenuse} = \frac{O}{H}$$

$$\text{Cos } \theta = \frac{adjacent}{hypotenuse} = \frac{A}{H}$$

$$\text{Tan } \theta = \frac{opposite}{adjacent} = \frac{O}{A}$$

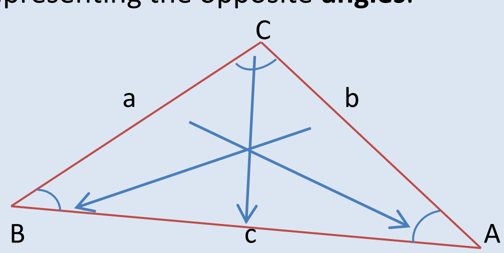


For a **non – right angle triangle**,
Use the **Sine/ Cosine Rule** to find unknown sides and angles.

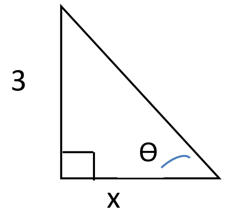
- SINE RULE $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ Or : $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

- COSINE RULE $a^2 = b^2 + c^2 - 2bc \text{ Cos A}$

Where **small letters** [a,b,c] represent the **sides**, and **capital letters** [A,B,C] are representing the opposite **angles**.



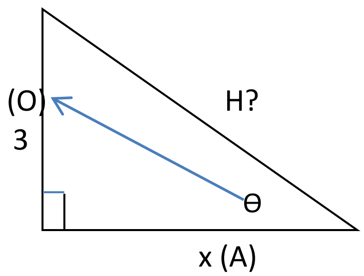
EXAMPLE 1: In the right triangle given below, find the value of *Cos Θ*?



Answer:

- Aim is to find the value of *Cos Θ,* Use 'CAH':

Cos θ = $\frac{adjacent}{hypotenuse} = \frac{A}{H}$ ➲ Label the A and H



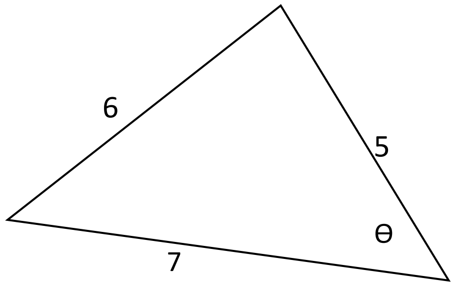
- A is marked 'x' and the expression for 'H' is needed in order to find Cos θ, thus use Pythagoras theorem

$$H^2 = a^2 + b^2$$
$$H^2 = x^2 + 3^2$$
$$H^2 = x^2 + 9$$
$$\sqrt{H^2} = \sqrt{x^2 + 9}$$
$$\therefore H = \sqrt{x^2 + 9}$$

Substitute in the expression:

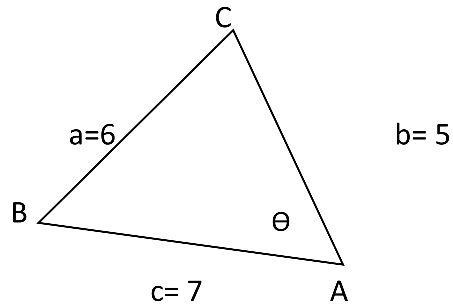
$$\boxed{\begin{aligned} Cos\theta &= \frac{A}{H} \\ \therefore Cos\theta &= \frac{x}{\sqrt{9 + x^2}} \end{aligned}}$$

EXAMPLE 2: Find the value for the angle Θ in the triangle shown below:



Answer:

Since only one angle given, let it be A: [Diagram not drawn to scale]



Use Cosine Rule to find the missing angle:

$a^2 = b^2 + c^2 – 2bc \text{ Cos } A$

$2bc \text{ Cos } A = b^2 + c^2 – a^2$ (take 2bc Cos A to the left and a^2 to the right by doing opposite operations)

$$\text{Cos } A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{Cos } A = \frac{5^2 + 7^2 - 6^2}{2 \times 5 \times 7} = \frac{38}{70}$$

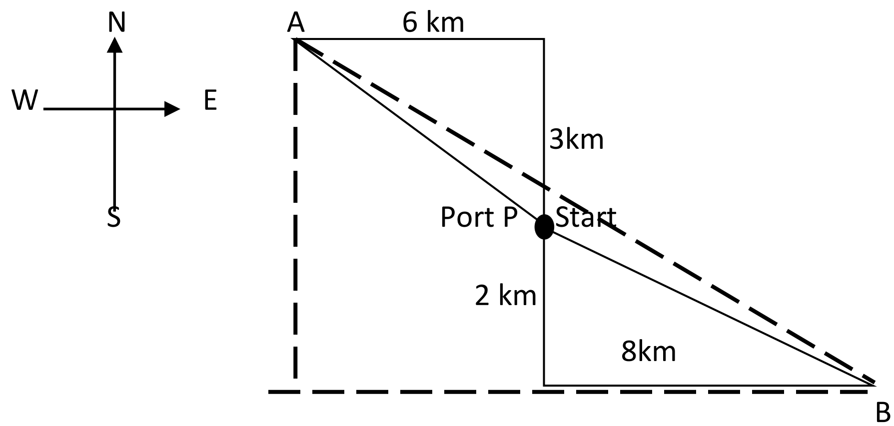
$$\text{Cos } A = \frac{38}{70}$$

To solve for A, do the opposite operation:

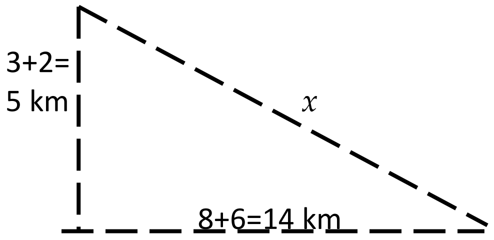
$$A = \text{Cos}^{-1}\left(\frac{38}{70}\right)$$

$$A = 57.12^o$$

EXAMPLE 3: Esekaia sails 3 km North and 6 km West from a Port P to destination A. Samit sails 2 km South and 8 km East from Port P to destination B. How far in a straight line is destination A from B?



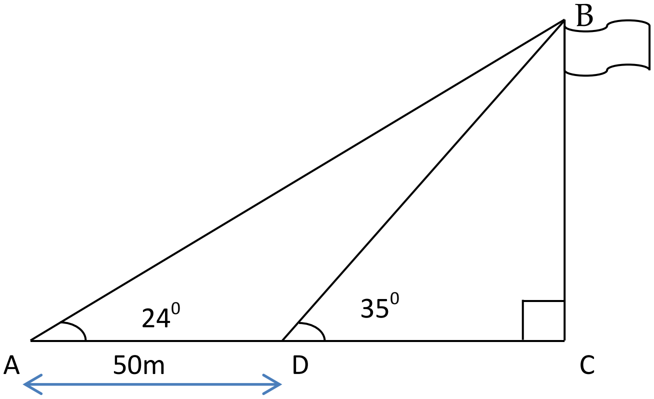
Answer:
Diagrammatically, we are interested in the bigger triangle,



Using Pythagoras theorem:

$$c^2 = a^2 + b^2$$
$$x^2 = 5^2 + 14^2$$
$$x^2 = 25 + 196$$
$$x^2 = 221$$
$$\therefore x = \sqrt{221} = 14.87km$$

EXAMPLE 4: The diagram below shows 2 angles of elevation of the flagpole, 24^0 and 35^0 at points A and D respectively. If point D is 50m from point A, find the distance $\overline{AB}$.

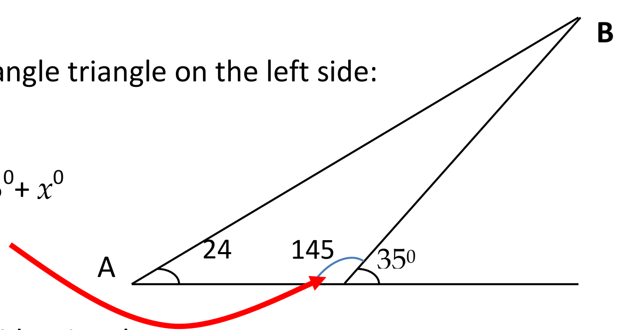


Answer:
Consider the non-right angle triangle on the left side:

Using straight lines,

$180^0 = 35^0 + x^0$

$x = 145^0$



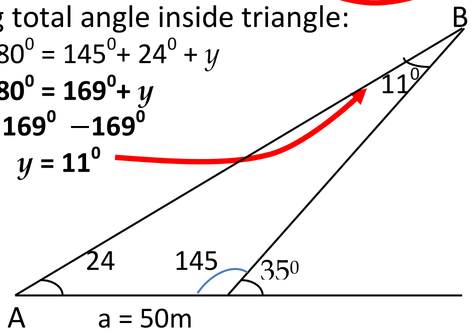
- Using total angle inside triangle:

  $180^0 = 145^0 + 24^0 + y$

  $\mathbf{180^0 = 169^0 + y}$

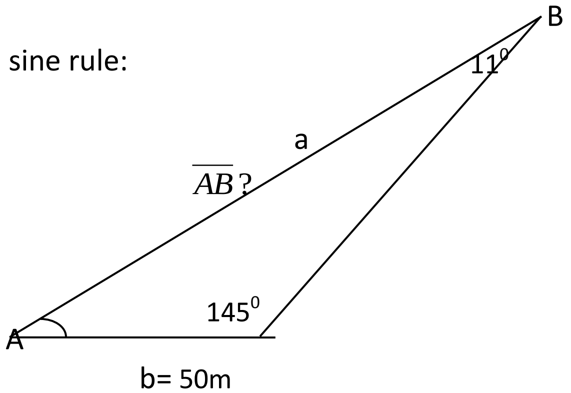
  $\mathbf{-169^0 \quad -169^0}$

  $\mathbf{y = 11^0}$



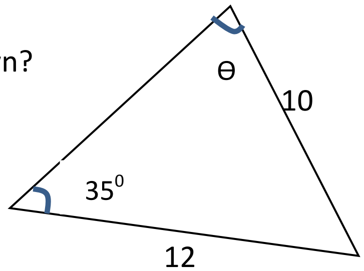
- Since we need to find the length $\overline{AB}$, use sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin \boldsymbol{B}}$$
$$\frac{\overline{AB}}{\sin 145} = \frac{50}{\sin 11} \text{ x } \sin 145$$
$$\frac{\overline{AB}}{\sin 145} \text{ x } \sin 145 = \frac{50}{\sin 11} \text{ x } \sin 145$$
$$\overline{AB} = \frac{50 \ \sin 145}{\sin 11}$$
$$\therefore \overline{AB} = 150.30\text{m}$$

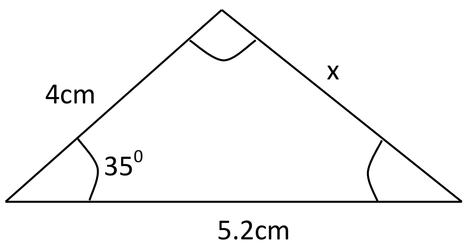


EXERCISE 40:

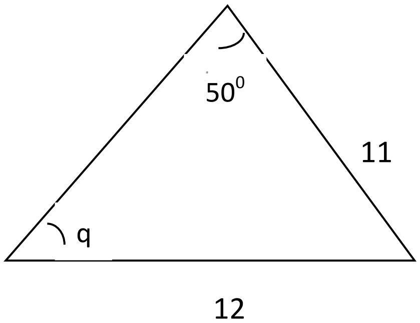
1. What is the value of ϴ in the triangle shown?



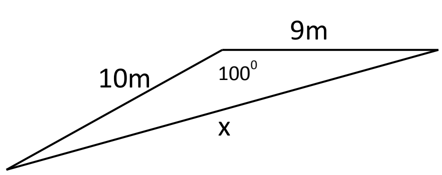
2. Find the value of x in the given triangle.



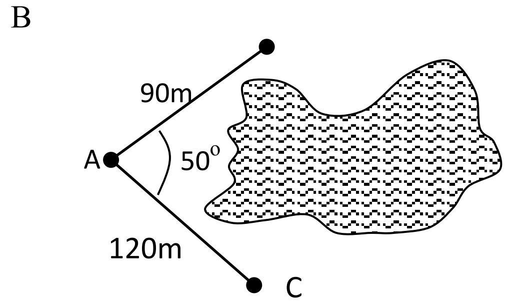
3. Find the value of angle marked a.



4. Use the cosine rule to solve for x.

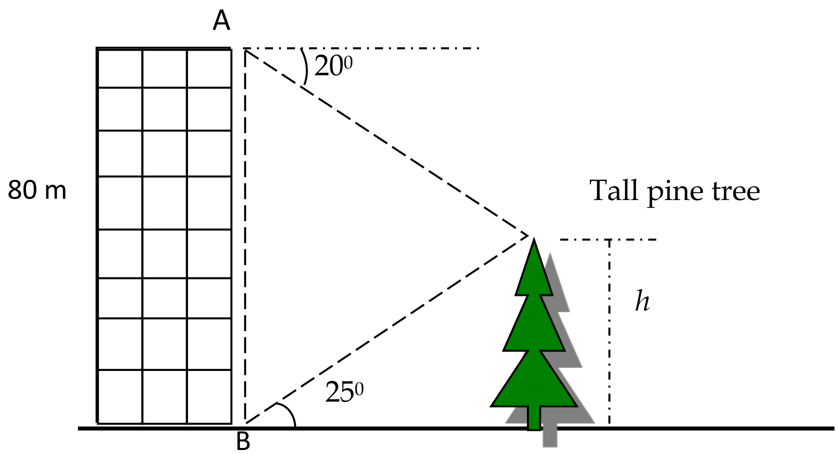


5. The distance between two points B and C on opposite sides of a lake is required. A surveyor locates a point A which is 90m from point B and 120m from point C. Angle BAC is 50^0. Calculate the length $\overline{BC}$.

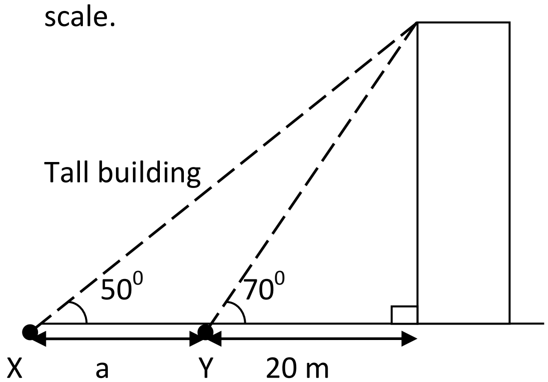


6. A private plane flies 1.3 hours at 110 mph on a bearing of 40°. Then it turns and continues another 1.5 hours at the same speed, but on a bearing of 130°. At the end of this time, how far is the plane from its starting point? What is its bearing from that starting point?

7. When the top of a tall pine tree is viewed from the top of a 8 – storey building (point A) 80 m above the ground, the angle of depression $= 20^0$ and when it is viewed from point B on the ground, the angle of elevation $= 25^0$. If points A and B are on the same vertical line, find *h*, the height of the tall pine tree. (Diagram not to scale)



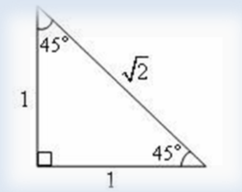
8. Rajjie is stationed at a Point Y, 20 m from the base of a tall building. He looks up to the top of the building at an angle of 70^0. Diagram not to scale.



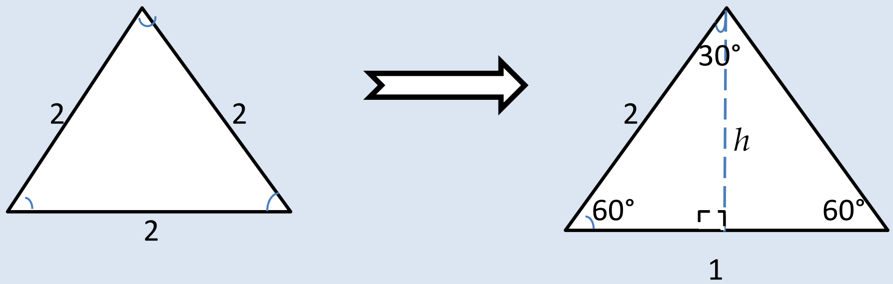
a) How high is the building?
b) Rajjie then moves back some distance so that he stands at Point X and now looks to the top of the building at an angle of 50^0. Calculate the distance 'a'.

FINDING EXACT VALUES OF SINE, COSINE AND TANGENT

There are two special triangles in trigonometry. One is the **(45°— 45°— 90°)** isosceles right triangle with both the base and height of length one unit.



The other is the triangle **30°- 60°- 90°** which is made by dividing an equilateral triangle of length 2 units into two halves.



Using the Pythagorean Theorem, the third side h yields:

$c^2 = a^2 + b^2$

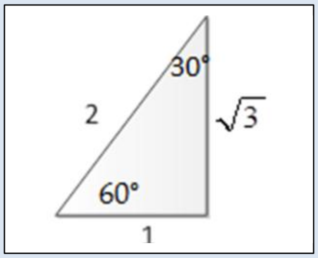
$2^2 = h^2 + 1^2$

$\therefore h^2 = 2^2 - 1^2$

$h^2 = 3$

$\sqrt{h^2} = \sqrt{3}$

$h = \sqrt{3}$



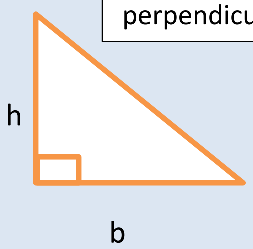
The table below shows the ratios of sides in a surd form:

Trig Function / Angles	**30°**	**45°**	**60°**	**90°**
Sin θ	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Cos θ	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Tan θ	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

AREA OF TRIANGLE

Right - angled triangle

Given Base and perpendicular Height

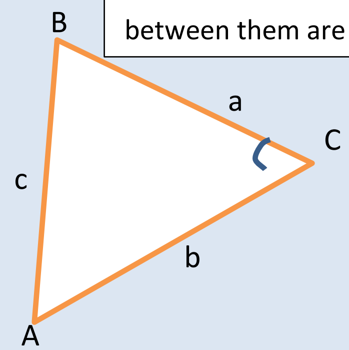


Area: $A = ½ bh$

Where b – base and
h – Perpendicular height

Non - right angled triangle

Two sides and the angle between them are known

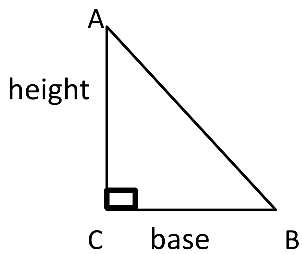


$A = ½ ab \sin C$ or

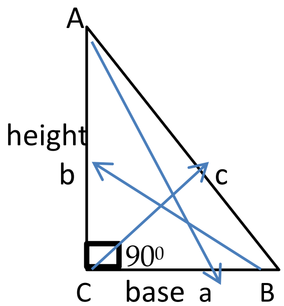
$A = ½ ac \sin B$ or $A = ½ bc \sin A$

Example 1: Show that the $A = ½ ab \sin C$ is equivalent to ½ base × height.

Consider a right angle triangle.



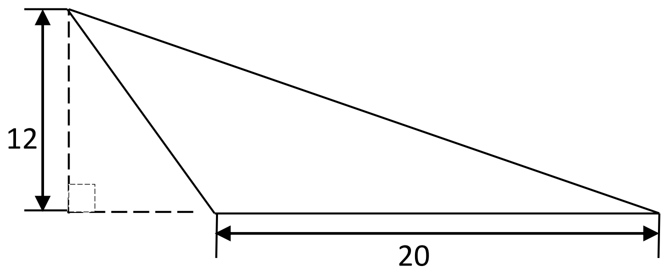
- Label the sides and angles:



- Use the formula for non – right angle triangle:

  Area $\Delta = ½ a \bullet b \sin C$
  $= ½ \text{base} \times \text{height} (\sin 90^0)$ ⮌ Since 90^0 is one
  $\therefore$ Area $\Delta = ½ \text{base} \times \text{height}$

Example 2: What is the area of the triangle given below?



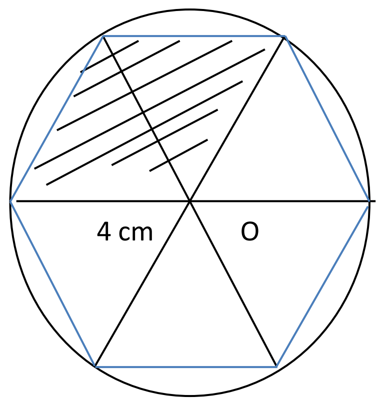
Answer:

Perpendicular Height: h = 12

Base: b = 20

Thus Area = **½ bh** = ½ × 20 × 12 = **120** cm^2

EXAMPLE 2: The diagram below shows a regular hexagon inscribed in a circle of radius 4 cm at centre O. (Diagram not to scale).



Calculate:

a) Angle of each sector formed.
b) the area of one of the triangle
c) the area of shaded region.

a) Angle region

$$T_{angle} = 360^0$$
$$6x = 360^0$$
$$x = 60^0$$

b) area of one of the triangle

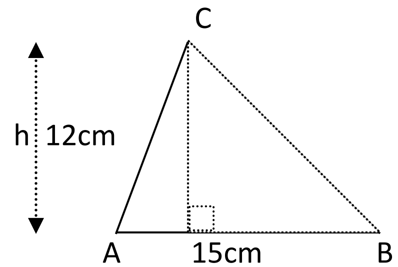
$$A_{\Delta} = \frac{1}{2}abSinC$$
$$= \frac{1}{2}(4 \times 4)Sin60^0$$
$$= 4\sqrt{3}cm^2$$

c) area of shaded

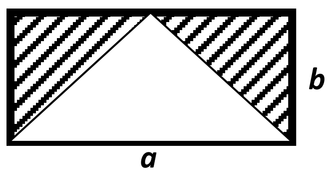
$$A = 2 \times A_{\Delta}$$
$$= 2 \times 4\sqrt{3}$$
$$= 8\sqrt{3}cm^2$$

EXERCISE 41:

1. Calculate the area of triangle ABC shown below.



2. Use the diagram given to answer the questions that follow.

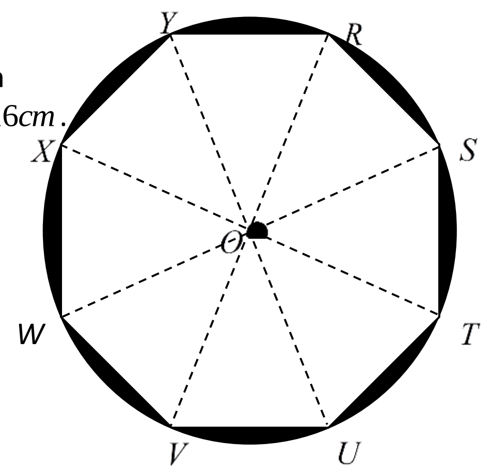


   a) Write an expression which could be used to calculate the area of the shaded region.
   b) If a = 120 cm and b = 40 cm, calculate the area of the shaded region.

3. The diagram below shows a regular octagon $RSTUVWXY$ inscribed in a circle of radius $16cm$. That is $\overline{OR} = 16cm$.

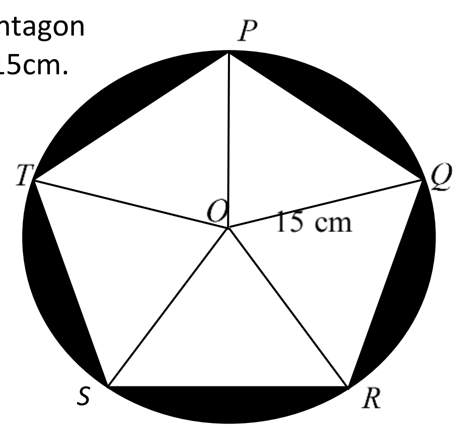
   a) Find the area of triangle ROS.

   b) Find the area of the octagon.



4. The diagram below shows a regular pentagon $PQRST$ inscribed in a circle of radius 15cm.

   a) Show that angle POQ is 72^0.
   b) Find the area of triangle POQ.
   c) Find the area of pentagon.



Lesson of Life: *Go down deep enough into anything and you will find mathematics* ~ Dean Schlicter

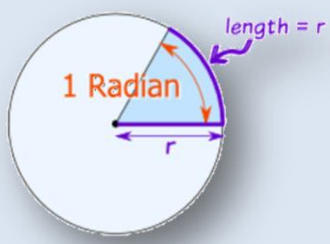
CONVERSION OF DEGREES TO RADIANS and VICE VERSA

Another unit for measuring angles is radians. Why learn radians? - What's wrong with degrees?
It used in many areas of mathematics. In science and engineering, **radians** are much more convenient (and common) than degrees. Many problems can be solved directly if we know the angle in radians especially in **calculus** (differentiation and integration).

How do we measure angles?

There are two main units of measuring angles: degrees (symbol '°') and radians.

One Radian: the angle made when we take the radius and wrap it along the edge of a circle.



One complete circle = 360° or 2π radians

- From above, π radians = 180°
  2π radians = 360°
- To convert degrees to radians: multiply by π/ 180
  $$\frac{\theta°}{180°} \times \pi = \text{angle in radians}$$
- To convert radians to degrees: divide by π/ 180 or multiply by 180 / π
  $$\text{Angle in radians} \times \frac{180°}{\pi} = \theta°$$

Conversion of some common angles

The table shows the conversion of some **common angles**.

Units	Values										
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
Degrees	30°	45°	60°	90°	120°	135°	180°	225°	270°	315°	360°

Example 1: Convert the following angles to radians

a) $45°$

$$\text{angle} = \frac{\theta°}{180°} \times \pi$$
$$= \frac{45°}{180°} \times \pi$$
$$= \frac{\pi}{4}$$

b) $15°$

$$\text{angle} = \frac{\theta°}{180°} \times \pi$$
$$= \frac{15°}{180°} \times \pi$$
$$= \frac{\pi}{12}$$

c) $25°$

$$\text{angle} = \frac{\theta°}{180°} \times \pi$$
$$= \frac{25°}{180°} \times \pi$$
$$= \frac{5\pi}{36}$$

Example 2: Convert the following angles to degrees

a) $\frac{\pi}{5}$

$$\text{angle} = \theta \times \frac{180°}{\pi}$$
$$= \frac{\pi}{5} \times \frac{180°}{\pi}$$
$$= 36°$$

b) $\frac{5\pi}{2}$

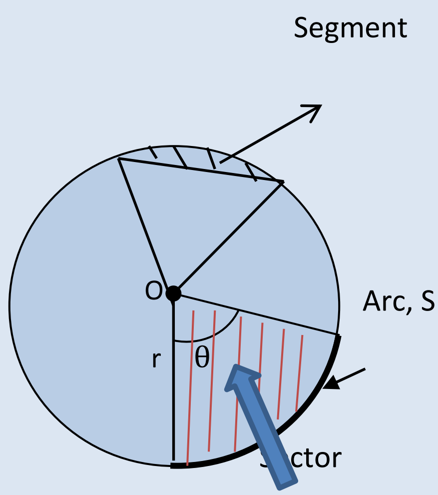
$$\text{angle} = \theta \times \frac{180°}{\pi}$$
$$= \frac{5\pi}{2} \times \frac{180°}{\pi}$$
$$= 450°$$

c) $\frac{4\pi}{7}$

$$\text{angle} = \theta \times \frac{180°}{\pi}$$
$$= \frac{4\pi}{7} \times \frac{180°}{\pi}$$
$$= \frac{720°}{7}$$
$$or 102.86°$$

ARC LENGTH, AREA OF SECTOR and SEGMENT

Note: Refer to the Parts of a circle



where r is the radius of the circle
θ is the angle in **radians**
O as **center** of circle

1. **ARC LENGTH (S)**
The arc length is the measure of the distance along the curved line making up the arc.

Formulae:

$$S = r\,\theta$$

[Angle must be in radians]

2. **AREA OF SECTOR**
Sector is the area enclosed by an arc and the two radii

Formulae:

$$\text{Area}_{\text{sector}} = \frac{\theta}{360^\circ} \times \pi r^2$$

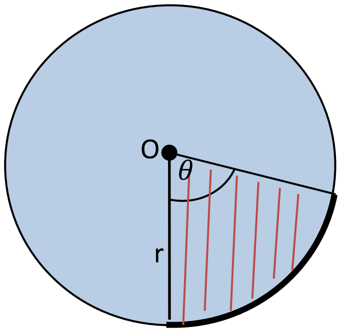
Since 360° = 2π radians , substituting yields

$$\text{Area}_{\text{sector}} = \frac{\theta}{2\pi} \times \pi r^2$$

$$= \frac{\theta}{2} \times r^2, \text{ rearranging}$$

$$= \tfrac{1}{2} r^2 \theta$$

[angle must be in radians]



3. **AREA OF SEGMENT**
Segment is the region bounded by a chord and an arc.

Formulae:

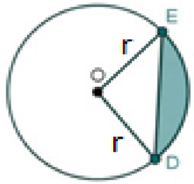
$$\text{Area}_{\text{segment}} = \text{Area}_{\text{sector}} - \text{Area}_{\text{triangle}}$$

$$= \tfrac{1}{2} r^2 \theta - \tfrac{1}{2} r^2 \sin\theta$$

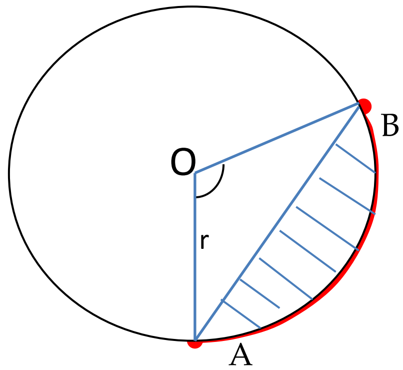
or

$$= \tfrac{1}{2} r^2 (\theta - \sin\theta)$$

[angle must be in radians]



Example 3: The diagram below shows a circle of radius r. OAB is a sector of the circle and has an angle of 95°.



i. Convert the angle to radians.
ii. Given that the segment has an area of 100 cm^2, calculate radius of the circle.
iii. Calculate the length of minor arc AB.
iv. Hence or otherwise, determine the perimeter of the sector.

Answers:

i. Conversion to radians

$$\text{angle} = \frac{\theta°}{180°} \times \pi$$

$$= \frac{95°}{180°} \times \pi$$

$$= \frac{19\pi}{36}$$

ii. radius of the circle

$$\text{Area}_{\text{segment}} = ½ r^2 (\theta - \sin\theta)$$

$$100 = ½ r^2 (\frac{19\pi}{36} - \sin\frac{19\pi}{36})$$

$$100 = 0.330934045\, r^2$$

$$\frac{100}{0.330934045} = r^2$$

$$r^2 = 302.175$$

$$r = 17.38 \text{ cm}$$

iii. Length of minor arc AB

$$S = r\theta$$

$$= 17.83 \times \frac{19\pi}{36}$$

$$= 28.82 \text{ cm}$$

iv. Perimeter of the sector: add all sides

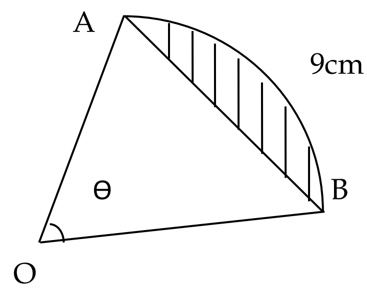
$$P = \overline{OA} + \overline{OB} + \text{Length of arc AB}$$

$$= 17.38 + 17.38 + 28.82$$

$$= 63.58 \text{ cm}$$

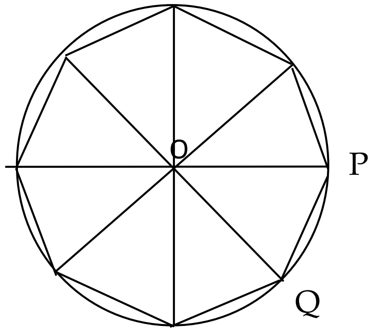
EXERCISE 42:

1. The diagram below shows a sector of a circle with the radius of 6cm and the length of the arc is 9cm.



a. Show that the angle ϴ = 1.5 rad.
b. Calculate the area of the sector OAB.
c. Calculate the area of the shaded segment.

2. The diagram below shows a regular octagon inscribed in a circle of radius 4 cm with centre O.

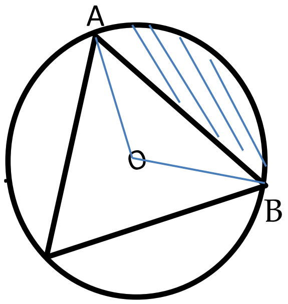


Calculate the following:

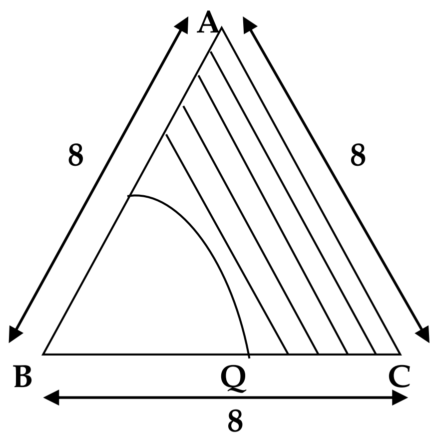
   a. Length of arc PQ
   b. Area of sector OPQ
   c. Area of triangle OPQ
   d. Area of the octagon

3. An equilateral triangle is inscribed in a circle with centre O. The radius of the circle is 5 cm.

   a. Find the area of the sector OAPB.
      P
   b. Find the area of the △ AOB
   c. Hence, calculate the area of the shaded region.



4.



In the diagram given above, P is the midpoint of AB; Q is the midpoint of BC

a. Find the length of the arc PQ.
b. Determine the area of the sector PBQ to the nearest whole number
c. Find the area of the shaded region.

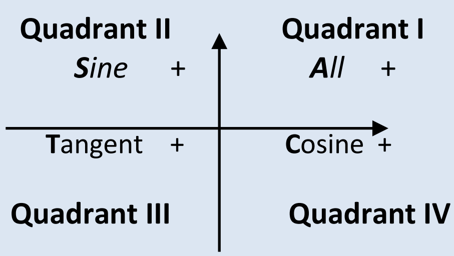
Lesson of Life: *The physical sciences (chemistry, physics, oceanography, astronomy) require mathematics for the development of their theories* - anonymous

SOLVING TRIGONOMETRIC EQUATIONS

Note:
When solving any trigonometric equation, emphasis must be given to the angle, θ, which can be either in degrees or radians.

To solve for θ, follow solving an algebraic equation:

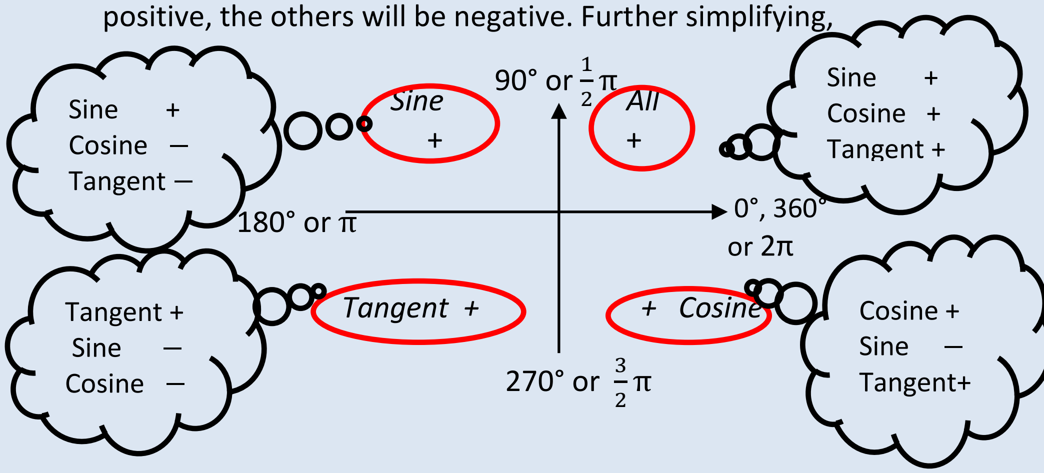
- The value consisting θ, to be removed last.
- Do opposite operation on both sides of the equation till you reach the trigonometric expression containing sine, cosines or tangent.
- At this point in time, keep in mind that there will be at least two angles, within 0^{o} to 360^{0} or 2π radians.
- If Trig expression is positive, then you will directly get the acute angle θ_1. If Trig expression is negative, then You may (ignore the negative sign to) get the acute angle α , use this to find the angle θ_1
- Use quadrants to find the other angle θ_2. Angles will be considered from the positive x – axis.



Mnemonic

All **S**cience **T**eachers **C**ry
or
Add **S**ugar **To** **C**offee

- If you look at the quadrants, the designated trig expressions will be positive, the others will be negative. Further simplifying,



Example 1: Solve $\tan\theta - 1 = 0,\ \ 0 \le \theta \le 2\pi$

Last

$\tan\theta - 1 = 0,$ $\quad 0 \le \theta \le 2\pi$ means that angle to be between 0 – 2π

$$\tan\theta \cancel{-1+1} = 0+1,$$
$$\tan\theta = 1$$
$$\theta_1 = \tan^{-1} 1$$
$$\theta_1 = 45^{\circ}$$

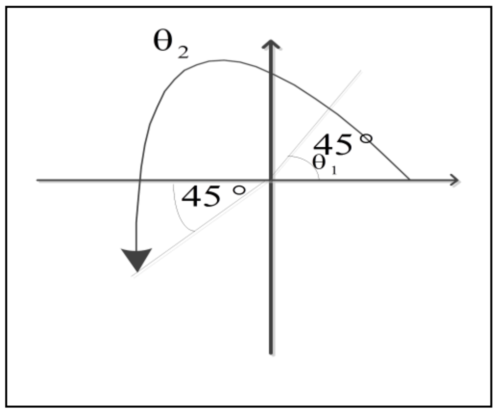
We reached at the trig expression: Consider the two quadrants. Find the acute angle in Q I. Note that calculator Mode to be in degrees.


Press Shift tan 1 =

- Use quadrants to find the angle θ_2. Consider sign (+) of 'tan', that is in Q III

$$\theta_2 = 180 + \theta_1$$
$$= 180 + 45$$
$$= 225^{o}$$

$$\boxed{\begin{array}{l}\theta = 45^{o}, 225^{o}\ or \\ \theta \in \{45^{o}, 225^{o}\}\end{array}}$$



Example 2: Find the solution set for $2Cos\theta + \sqrt{3} = 0, 0^{o} \le \theta \le 360^{0}$

$2Cos\theta + \sqrt{3} = 0,$ $\quad 0^{o} \le \theta \le 360^{0}$ Means that angle to be between 0 – 360°

Last

$$2Cos\theta + \sqrt{3} \cancel{- \sqrt{3}} = 0 - \sqrt{3}$$

$$\frac{\cancel{2}Cos\theta}{\cancel{2}} = \frac{-\sqrt{3}}{2}$$

$$Cos\theta = \frac{-\sqrt{3}}{2}$$

We reached at the trig expression: Consider the two quadrants. But before that, find the acute angle by ignoring the negative sign (–). Note that calculator Mode to be in degrees.

- **Acute angle:**

$$\theta = Cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^o$$

$$\alpha = 30^o$$

When dealing with surds,

press the division sign (÷), that is Press

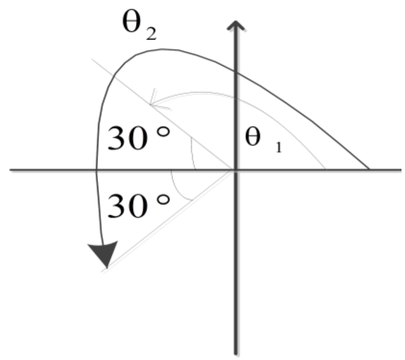
Shift | cos | (| $\sqrt{3}$ | ÷ | 2 |) | =

- Use quadrants to find the angles θ_1 and θ_2. Consider negative sign (–) of Cos, that is in Q II / III

$\theta_1 = 180 - 30 = 150^o$

$\theta_2 = 180 + 30 = 210^o$

$$\boxed{\begin{array}{l}\theta = 150^o, 210^o \; or \\ \theta \in \{150^o, 210^o\}\end{array}}$$



EXAMPLE 3: Solve the trigonometric equation $\sin(x + 30^0) = 0.4$, where $-180^0 \le x \le 180^0$.

Acute angle:

It already has trig expression:
Consider the two quadrants. Note that calculator Mode to be in degrees.

$\sin(x + 30^0) = 0.4$

$(x + 30^0) = \sin^{-1} 0.4$

$\alpha = 23.58^o$

Press

Shift | sin | 0 | . | 4 | =

- Use quadrants to find the angles θ_1 and θ_2. Consider positive sign (+) of sin, that is in Q I / II

$\theta_2 = 180 - \alpha = 180 - 23.58 = 156.42^o$ $\alpha = \theta_1 = 23.58^o$

$QI \; \& \; II:$

$QI: x + 30^0 = 23.58,$

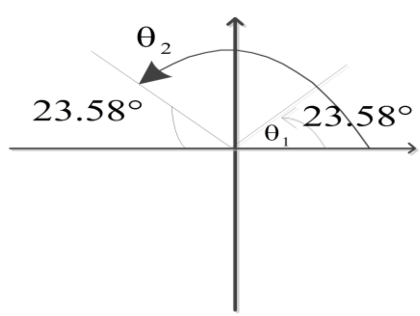
$x + 30^0 = 23.58,$

$x = -6.42^o$

$QII: x + 30^0 = 156.42$

$x = 126.42^o$

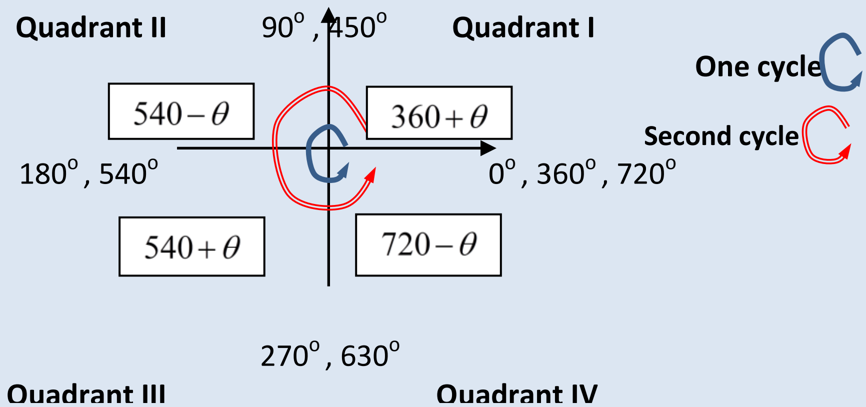
$\therefore x \in \{-6.42^0, 126.42^0\}$



So far the coefficient of the variable was one. If the coefficient changes to other number, let's then expect the solution to double, that is

$1x$ ➲ two solutions
$2x$ ➲ four solutions
$3x$ ➲ six solutions and so on.

For the first cycle, angle is found the normal way as before.
For the second cycle, the angles can be found as:



EXAMPLE 4: Solve $2\cos 2x = \sqrt{3}$ for $0 \le x \le 2\pi$

$2\cos 2x = \sqrt{3}$ $\quad 0 \le x \le 2\pi$ Angle to be between $0 - 360^o$

Last

$\frac{\cancel{2}\cos 2x}{\cancel{2}} = \frac{\sqrt{3}}{2}$

$Cos 2x = \frac{\sqrt{3}}{2}$

We reached at the trig expression and the value in front of the variable is 2, so it has 4 solutions. If you find easier in degrees, you may do so. Once all solutions are found, then you can convert to radians as the question requires you to give answers in radians.

Acute angle:

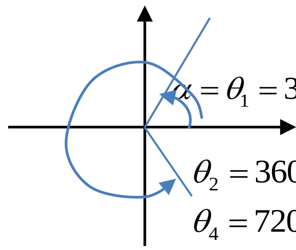
$2x = Cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^o$

$\alpha = 30^o$


(when dealing with surds, press the division sign (÷), that is Press

$\boxed{\text{Shi}}$ $\boxed{\text{co}}$ $\boxed{(}$ $\boxed{\sqrt{3}}$ $\boxed{\div}$ $\boxed{2}$ $\boxed{)}$ $\boxed{=}$

- Use quadrants to find the **four** angles: Cos is positive in Quadrants I / IV



$\alpha = \theta_1 = 30^o$, $\theta_3 = 360 + \alpha = 360 + 30 = 390^o$

$\theta_2 = 360 - \alpha = 360 - 30 = 330^o$

$\theta_4 = 720 - \alpha = 720 - 30 = 690^o$

$$2x = \{30^o, 330^o, 390^o, 690^o\}$$

Divide by 2, to all the angles

$$\frac{2x}{2} = \left\{\frac{30^o}{2}, \frac{330^o}{2}, \frac{390^o}{2}, \frac{690^o}{2}\right\}$$

$$x = \{15^o, 165^o, 195^o, 345^o\}$$

To convert in radians, divide by 180 and multiply by π :

$$x = \left\{\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}\right\}$$

EXERCISE 43: Solve

1. $\sin(x - \frac{\pi}{4}) = 0$, for $0 \le x \le 2\pi$.

2. $2 \sin x + \sqrt{3} = 0$, for $0^0 \le x \le 360^0$.

3. $\cos 2x = \frac{1}{2}$, for $0^0 \le x \le 360^0$.

4. $2 \cos(\theta - \frac{\pi}{4}) = 1$ for $0 \le \theta \le 2\pi$.

5. $\tan 2x - 1 = 0$, for $0^0 \le x \le 360^0$

6. $\text{Sin}(2x + \frac{\pi}{4})$ for $0 \le x \le 2\pi$.

Lesson of Life: *If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is*. ~John Louis von Neumann