CHAPTER 7

Euclidean geometry

7 Euclidean geometry

Geometry (from the Greek "geo" = earth and "metria" = measure) arose as the field of knowledge dealing with spatial relationships. Analytical geometry deals with space and shape using algebra and a coordinate system. Euclidean geometry deals with space and shape using a system of logical deductions.

Euclidean geometry was first used in surveying and is still used extensively for surveying today. Euclidean geometry is also used in architecture to design new buildings. Other uses of Euclidean geometry are in art and to determine the best packing arrangement for various types of objects.

Figure 7.1: A small piece of the original version of Euclid's elements. Euclid is considered to be the father of modern geometry. Euclid's elements was used for many years as the standard text for geometry.

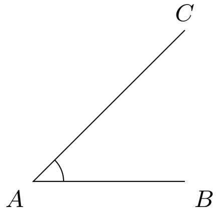
VISIT:
This video highlights some of the basic concepts used in geometry.
See video: 2G5V at www.everythingmaths.co.za

DID YOU KNOW?
In Euclidean geometry we use two fundamental types of measurement: angles and distances.

7.1 Introduction EMA5M

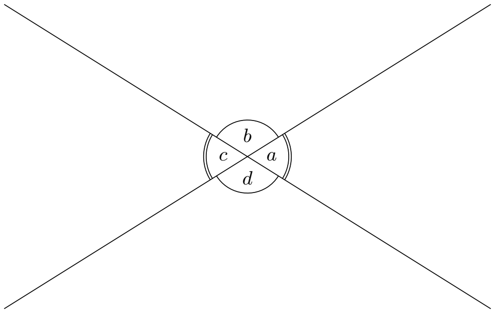
Angles EMA5N

An angle is formed when two straight lines meet at a point, also known as a vertex. Angles are labelled with a caret on a letter, for example, $\hat{B}$. Angles can also be labelled according to the line segments that make up the angle, for example $C\hat{B}A$ or $A\hat{B}C$. The $\angle$ symbol is a short method of writing angle in geometry and is often used in phrases such as "sum of $\angle$s in $\triangle$". Angles are measured in degrees which is denoted by °, a small circle raised above the text, similar to an exponent.



Properties and notation EMA5P

In the diagram below two straight lines intersect at a point, forming the four angles $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$.



The following table summarises the different types of angles, with examples from the figure above.

Term	Property	Examples
Acute angle	$0° < \text{angle} < 90°$	$\hat{a}$; $\hat{c}$
Right angle	Angle $= 90°$	
Obtuse angle	$90° < \text{angle} < 180°$	$\hat{b}$; $\hat{d}$
Straight angle	Angle $= 180°$	$\hat{a}+\hat{b}$; $\hat{b}+\hat{c}$
Reflex angle	$180° < \text{angle} < 360°$	$\hat{a}+\hat{b}+\hat{c}$
Adjacent angles	Angles that share a vertex and a common side.	$\hat{a}$ and $\hat{d}$; $\hat{c}$ and $\hat{d}$
Vertically opposite angles	Angles opposite each other when two lines intersect. They share a vertex and are equal.	$\hat{a}=\hat{c}$; $\hat{b}=\hat{d}$
Supplementary angles	Two angles that add up to $180°$	$\hat{a}+\hat{b}=180°$; $\hat{b}+\hat{c}=180°$
Complementary angles	Two angles that add up to $90°$	
Revolution	The sum of all angles around a point.	$\hat{a}+\hat{b}+\hat{c}+\hat{d}=360°$

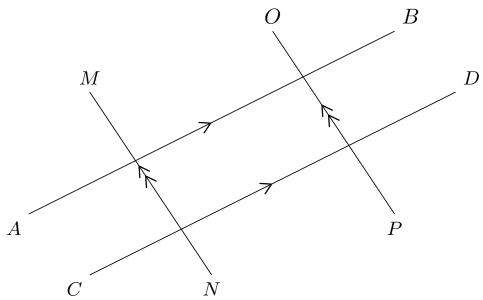
Note that adjacent angles on a straight line are supplementary.

VISIT:
The following video provides a summary of the terms used to refer to angles.
See video: 2G5W at www.everythingmaths.co.za

Parallel lines and transversal lines EMA5Q

Two lines intersect if they cross each other at a point. For example, at a traffic intersection two or more streets intersect; the middle of the intersection is the common point between the streets.

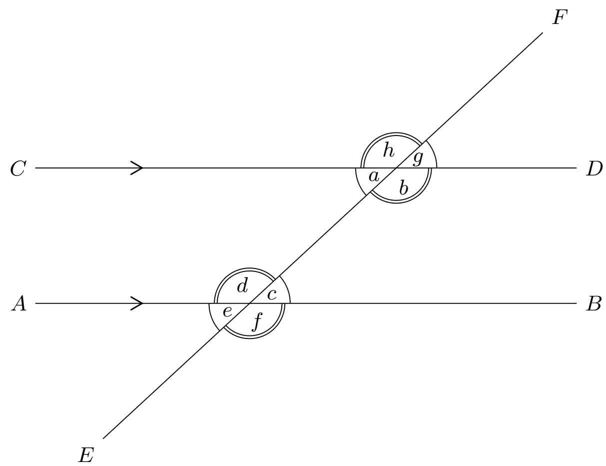
Parallel lines are always the same distance apart and they are denoted by arrow symbols as shown below.



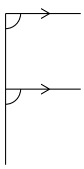
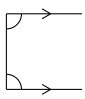
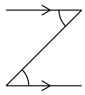
In writing we use two vertical lines to indicate that two lines are parallel:

$$AB \parallel CD \text{ and } MN \parallel OP$$

A transversal line intersects two or more parallel lines. In the diagram below, $AB \parallel CD$ and EF is a transversal line.



The properties of the angles formed by these intersecting lines are summarised in the following table:

Name of angle	Definition	Examples	Notes
Interior angles	Angles that lie in between the parallel lines.	$\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are interior angles.	Interior means inside.
Exterior angles	Angles that lie outside the parallel lines.	$\hat{e}$, $\hat{f}$, $\hat{g}$ and $\hat{h}$ are exterior angles.	Exterior means outside.
Corresponding angles	Angles on the same side of the lines and the same side of the transversal. If the lines are parallel, the corresponding angles will be equal.	$\hat{a}$ and $\hat{e}$, $\hat{b}$ and $\hat{f}$, $\hat{c}$ and $\hat{g}$, $\hat{d}$ and $\hat{h}$ are pairs of corresponding angles. $\hat{a} = \hat{e}$, $\hat{b} = \hat{f}$, $\hat{c} = \hat{g}$ and $\hat{d} = \hat{h}$.	 F shape
Co-interior angles	Angles that lie in between the lines and on the same side of the transversal. If the lines are parallel, the angles are supplementary.	$\hat{a}$ and $\hat{d}$, $\hat{b}$ and $\hat{c}$ are pairs of co-interior angles. $\hat{a} + \hat{d} = 180°$, $\hat{b} + \hat{c} = 180°$.	 C shape
Alternate interior angles	Equal interior angles that lie inside the lines and on opposite sides of the transversal. If the lines are parallel, the interior angles will be equal.	$\hat{a}$ and $\hat{c}$, $\hat{b}$ and $\hat{d}$ are pairs of alternate interior angles. $\hat{a} = \hat{c}$, $\hat{b} = \hat{d}$	 Z shape

VISIT:
This video provides a short summary of some of the angles formed by intersecting lines.
See video: 2G5X at www.everythingmaths.co.za

If two lines are intersected by a transversal such that:

- corresponding angles are equal; or
- alternate interior angles are equal; or
- co-interior angles are supplementary

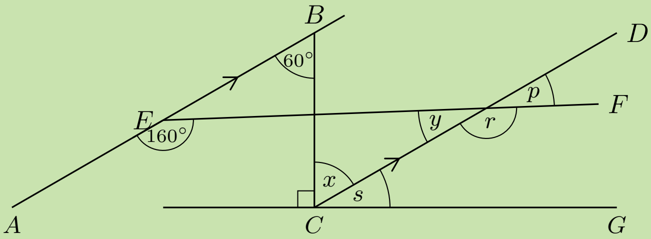
then the two lines are parallel.

NOTE:
When we refer to lines we can either write EF to mean the line through points E and F or $\overline{EF}$ to mean the line segment from point E to point F.

Worked example 1: Finding angles

QUESTION

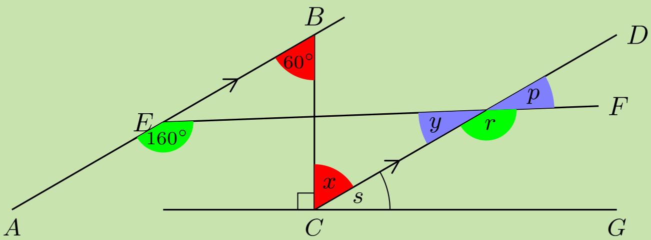
Find all the unknown angles. Is $EF \parallel CG$? Explain your answer.



SOLUTION

Step 1: Use the properties of parallel lines to find all equal angles on the diagram

Redraw the diagram and mark all the equal angles.



Step 2: Determine the unknown angles

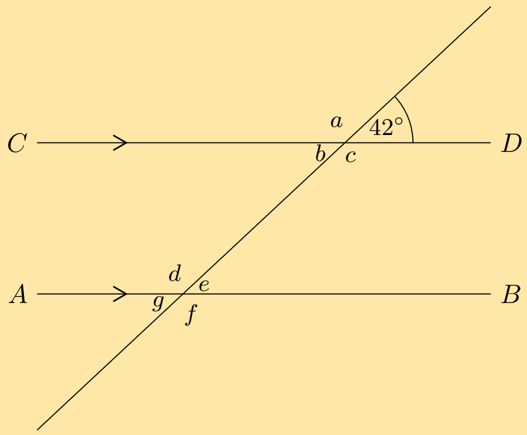
$$
\begin{aligned}
AB &\parallel CD && \text{(given)}\\
\therefore \hat{x} &= 60^\circ && \text{(alt } \angle\text{s; } AB \parallel CD)\\
\hat{y} + 160^\circ &= 180^\circ && \text{(co-int } \angle\text{s; } AB \parallel CD)\\
\therefore \hat{y} &= 20^\circ && \\
\hat{p} &= \hat{y} && \text{(vert opp } \angle\text{s } =)\\
\therefore \hat{p} &= 20^\circ && \\
\hat{r} &= 160^\circ && \text{(corresp } \angle\text{s; } AB \parallel CD)\\
\hat{s} + \hat{x} + 90^\circ &= 180^\circ && (\angle\text{s on a str line})\\
\hat{s} + 60^\circ &= 90^\circ && \\
\therefore \hat{s} &; = 30^\circ &&
\end{aligned}
$$

Step 3: Determine whether $EF \parallel CG$

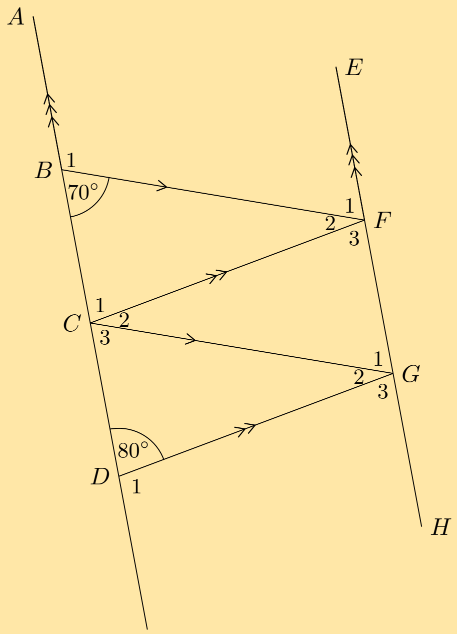
If $EF \parallel CG$ then $\hat{p}$ will be equal to corresponding angle $\hat{s}$, but $\hat{p} = 20^\circ$ and $\hat{s} = 30^\circ$. Therefore EF is not parallel to CG.

Exercise 7 – 1:

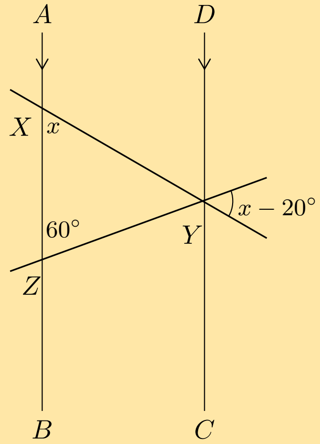
1. Use adjacent, corresponding, co-interior and alternate angles to fill in all the angles labelled with letters in the diagram:



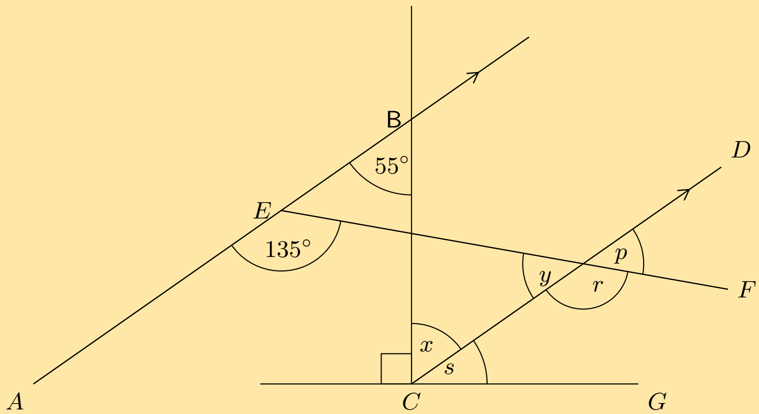
2. Find all the unknown angles in the figure:



3. Find the value of x in the figure:

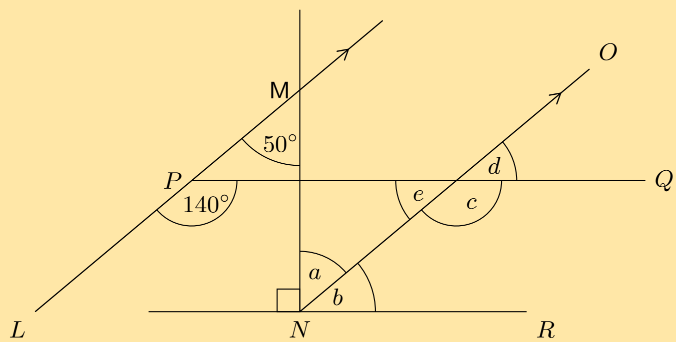


4. Given the figure below:



a) Find each of the unknown angles marked in the figure below. Find a reason that leads to the answer in a single step.
b) Based on the results for the angles above, is $EF \parallel CG$?

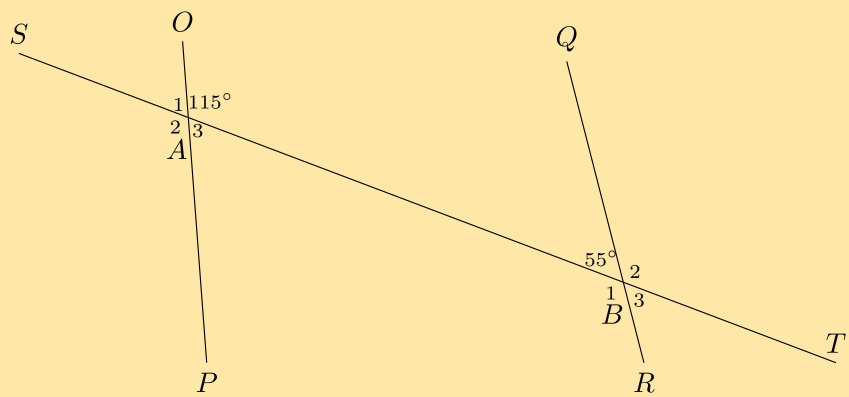
5. Given the figure below:



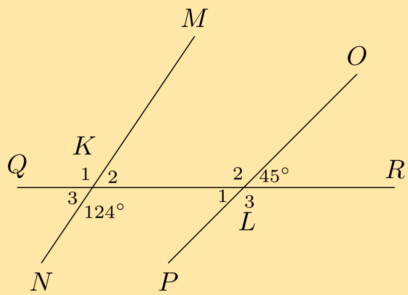
a) Find each of the unknown angles marked in the figure below. Find a reason that leads to the answer in a single step.
b) Based on the results for the angles above, is $PQ \parallel NR$?

6. Determine whether the pairs of lines in the following figures are parallel:

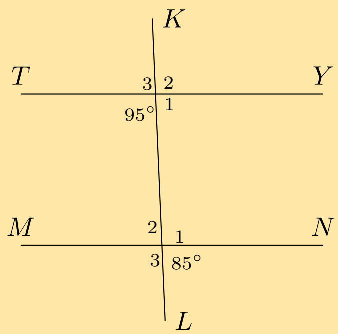
a)



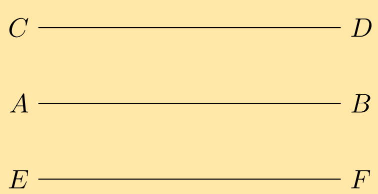
b)



c)



7. If AB is parallel to CD and AB is parallel to EF, explain why CD must be parallel to EF.



For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2G5Y 2. 2G5Z 3. 2G62 4. 2G63 5. 2G64 6a. 2G65 6b. 2G66 6c. 2G67 7. 2G68

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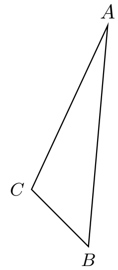
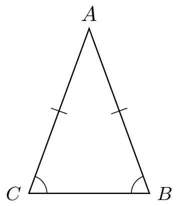
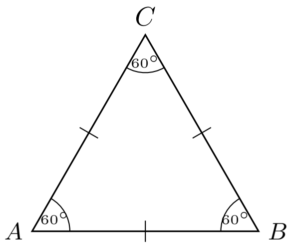
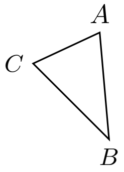
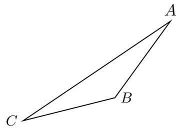
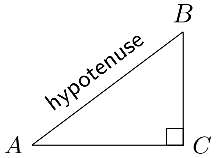
7.2 Triangles EMA5R

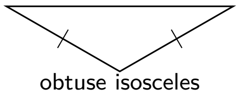
Classification of triangles EMA5S

A triangle is a three-sided polygon. Triangles can be classified according to sides: equilateral, isosceles and scalene. Triangles can also be classified according to angles: acute-angled, obtuse-angled and right-angled.

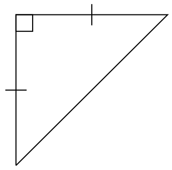
We use the notation $\triangle ABC$ to refer to a triangle with vertices labelled A, B and C.

Name	Diagram	Properties
Scalene		All sides and angles are different.
Isosceles		Two sides are equal in length. The angles opposite the equal sides are also equal.
Equilateral		All three sides are equal in length and all three angles are equal.
Acute		Each of the three interior angles is less than $90°$.
Obtuse		One interior angle is greater than $90°$.
Right-angled		One interior angle is $90°$.

Different combinations of these properties are also possible. For example, an obtuse isosceles triangle and a right-angled isosceles triangle are shown below:



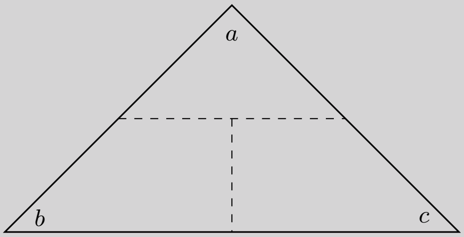
obtuse isosceles



right-angled isosceles

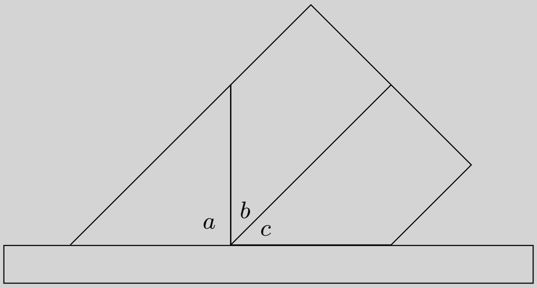
VISIT:
This video shows the different ways to classify triangles.
See video: 2G69 at www.everythingmaths.co.za

Investigation: Interior angles of a triangle

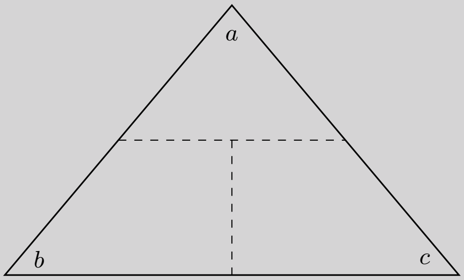


1. On a piece of paper draw a triangle of any size and shape.
2. Cut it out and label the angles $\hat{a}$, $\hat{b}$ and $\hat{c}$ on both sides of the paper.
3. Draw dotted lines as shown and cut along these lines to get three pieces of paper.
4. Place them along your ruler as shown in the figure below.
5. What can we conclude?

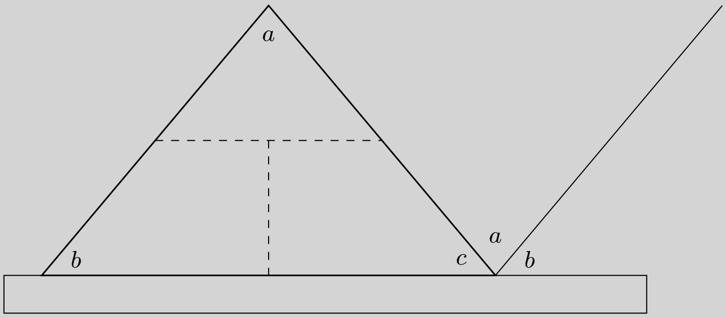
Hint: What is the sum of angles on a straight line?



Investigation: Exterior angles of a triangle



1. On a piece of paper draw a triangle of any size and shape. On another piece of paper, make a copy of the triangle.
2. Cut both out and label the angles of both triangles $\hat{a}$, $\hat{b}$ and $\hat{c}$ on both sides of the paper.
3. Draw dotted lines on **one** triangle as shown and cut along the lines.
4. Place the second triangle and the cut out pieces as shown in the figure below.
5. What can we can conclude?

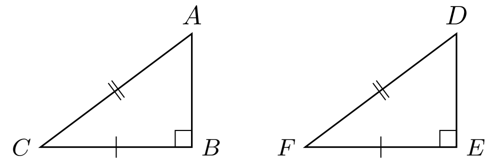
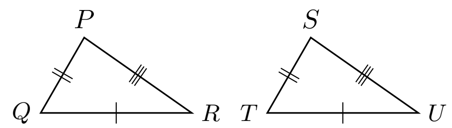
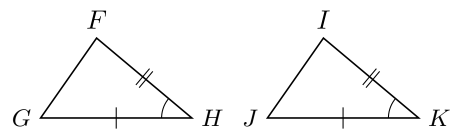
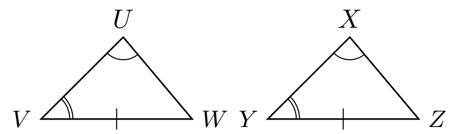


VISIT:
We can use the fact that the angles in a triangle add up to $180°$ to work out the sum of the exterior angles in a pentagon. This video shows you how.
See video: 2G6B at www.everythingmaths.co.za

Congruency EMA5T

Two triangles are congruent if one fits exactly over the other. This means that the triangles have equal corresponding angles and sides. To determine whether two triangles are congruent, it is not necessary to check every side and every angle. We indicate congruency using $\equiv$.

The following table describes the requirements for congruency:

Rule	Description	Diagram
RHS or 90°HS (90°, hypotenuse, side)	If the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and the corresponding side of another right-angled triangle, then the two triangles are congruent.	 $\triangle ABC \equiv \triangle DEF$
SSS (side, side, side)	If three sides of a triangle are equal in length to the corresponding sides of another triangle, then the two triangles are congruent.	 $\triangle PQR \equiv \triangle STU$
SAS or S∠S (side, angle, side)	If two sides and the **included** angle of a triangle are equal to the corresponding two sides and **included** angle of another triangle, then the two triangles are congruent.	 $\triangle FGH \equiv \triangle IJK$
AAS or ∠∠S (angle, angle, side)	If one side and two angles of a triangle are equal to the corresponding one side and two angles of another triangle, then the two triangles are congruent.	 $\triangle UVW \equiv \triangle XYZ$

The order of letters when labelling congruent triangles is very important.

$$\triangle ABC \equiv \triangle DEF$$

This notation indicates the following properties of the two triangles: $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$, $\hat{C} = \hat{F}$, $AB = DE$, $AC = DF$ and $BC = EF$.

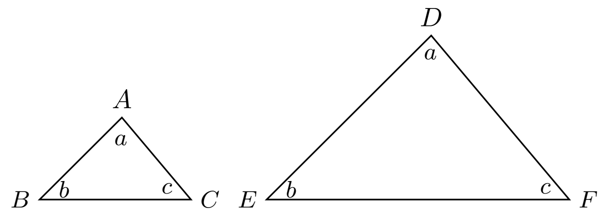
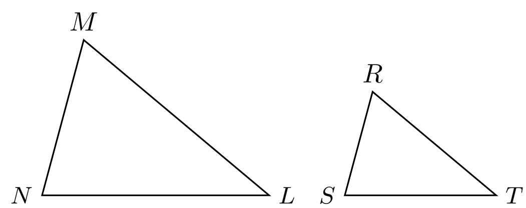
NOTE:
You might see $\cong$ used to show that two triangles are congruent. This is the internationally recognised symbol for congruency.

VISIT:
This video shows some practice examples of finding congruent triangles.
See video: 2G6C at www.everythingmaths.co.za

Similarity

Two triangles are similar if one triangle is a scaled version of the other. This means that their corresponding angles are equal in measure and the ratio of their corresponding sides are in proportion. The two triangles have the same shape, but different scales. Congruent triangles are similar triangles, but not all similar triangles are congruent. We use $|||$ to indicate that two triangles are similar.

The following table describes the requirements for similarity:

Rule	Description	Diagram
AAA (angle, angle, angle)	If all three pairs of corresponding angles of two triangles are equal, then the triangles are similar.	 $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$, $\hat{C} = \hat{F}$ $\therefore \triangle ABC \ \||| \ \triangle DEF$
SSS (side, side, side)	If all three pairs of corresponding sides of two triangles are in proportion, then the triangles are similar.	 $\frac{MN}{RS} = \frac{ML}{RT} = \frac{NL}{ST}$ $\therefore \triangle MNL \ \||| \ \triangle RST$

The order of letters for similar triangles is very important. Always label similar triangles in corresponding order. For example,

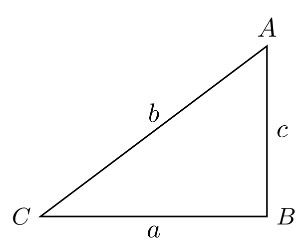
$\triangle MNL \ ||| \ \triangle RST$ is correct; but

$\triangle MNL \ ||| \ \triangle RTS$ is incorrect.

NOTE:
You might see $\sim$ used to show that two triangles are similar. This is the internationally recognised symbol for similarity.

VISIT:
The following video explains similar triangles.
See video: 2G6D at www.everythingmaths.co.za

The theorem of Pythagoras



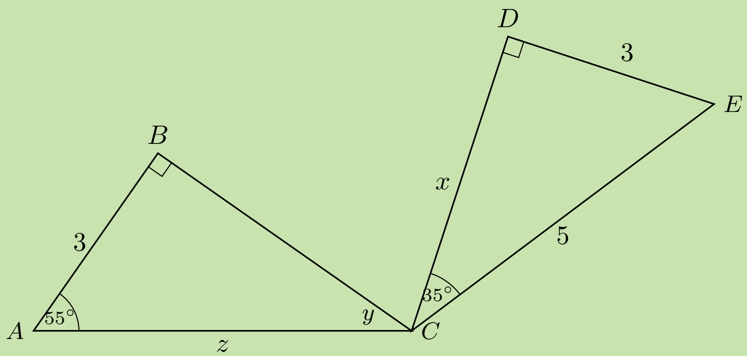
If $\triangle ABC$ is right-angled with $\hat{B} = 90°$, then $b^2 = a^2 + c^2$.
Converse: If $b^2 = a^2 + c^2$, then $\triangle ABC$ is right-angled with $\hat{B} = 90°$.

VISIT:
The following video explains the theorem of Pythagoras and shows some examples of working with the theorem of Pythagoras.
See video: 2G6F at www.everythingmaths.co.za

Worked example 2: Triangles

QUESTION

Determine if the two triangles are congruent. Use the result to find x, $\hat{y}$ and z.



SOLUTION

Step 1: Examine the information given for both triangles

Step 2: Determine whether $\triangle CDE \equiv \triangle CBA$

In $\triangle CDE$:

$$\hat{D} + \hat{C} + \hat{E} = 180° \quad \text{(sum of } \angle\text{s in } \triangle\text{)}$$
$$90° + 35° + \hat{E} = 180°$$
$$\therefore \hat{E} = 55°$$

In $\triangle CDE$ and $\triangle CBA$:

$$\begin{aligned} D\hat{E}C &= B\hat{A}C = 55° && \text{(proved)} \\ C\hat{D}E &= C\hat{B}A = 90° && \text{(given)} \\ DE &= BA = 3 && \text{(given)} \\ \therefore \triangle CDE &\equiv \triangle CBA && \text{(AAS)} \end{aligned}$$

Step 3: Determine the unknown angles and sides

In $\triangle CDE$:

$$CE^2 = DE^2 + CD^2 \quad \text{(Pythagoras)}$$
$$5^2 = 3^2 + x^2$$
$$x^2 = 16$$
$$\therefore x = 4$$

In $\triangle CBA$:

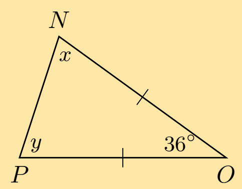
$$\begin{aligned} \hat{B} + \hat{A} + \hat{y} &= 180° && \text{(sum of } \angle\text{s in } \triangle) \\ 90° + 55° + \hat{y} &= 180° \\ \therefore \hat{y} &= 35° \end{aligned}$$

$$\begin{aligned} \triangle CDE &\equiv \triangle CBA && \text{(proved)} \\ \therefore CE &= CA \\ \therefore z &= 5 \end{aligned}$$

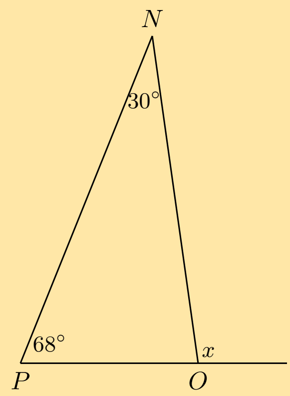
Exercise 7 – 2:

1. Calculate the unknown variables in each of the following figures.

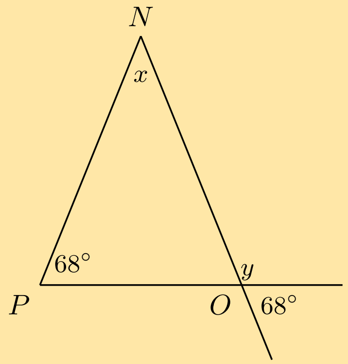
a)



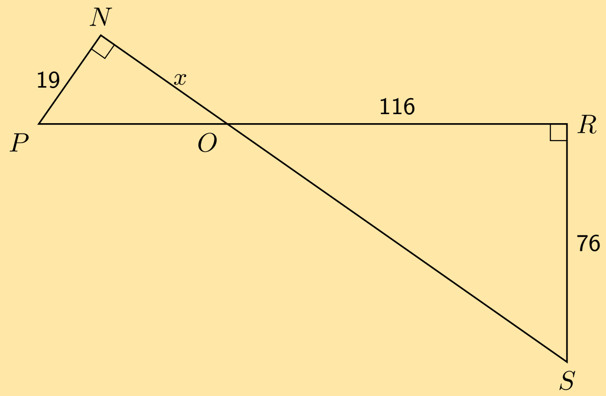
b)



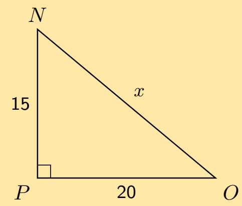
c)



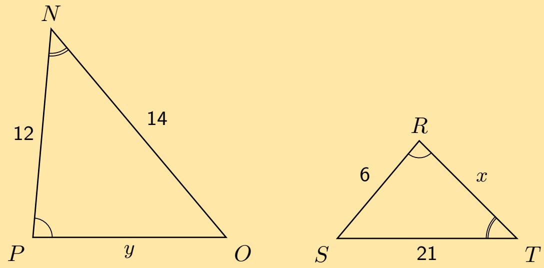
d)



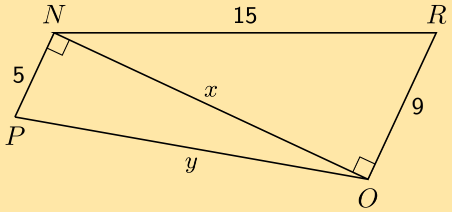
e)



f)



g)



2. Given the following diagrams:

   Diagram A

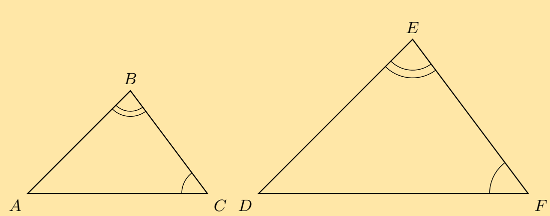
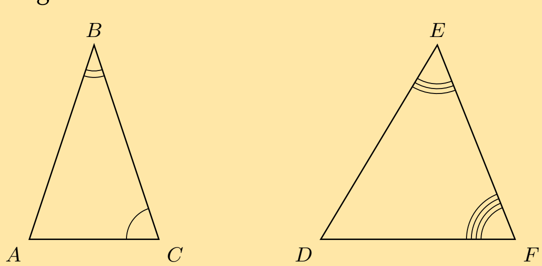
   

   Diagram B

   Which diagram correctly gives a pair of similar triangles?

3. Given the following diagrams:

Diagram A

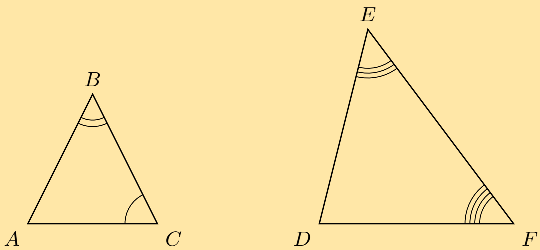
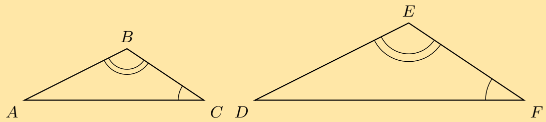
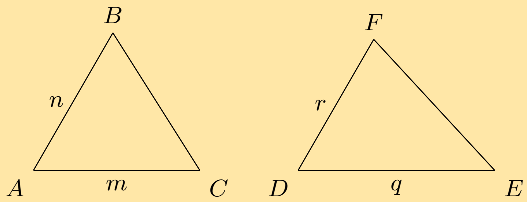


Diagram B



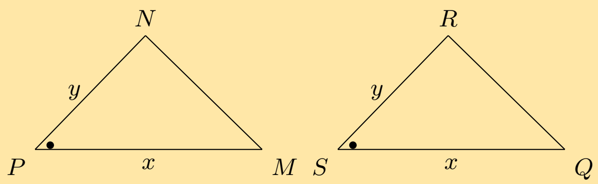
Which diagram correctly gives a pair of similar triangles?

4. Have a look at the following triangles, which are drawn to scale:



Are the two triangles congruent? If so state the reason and use the correct notation to state that they are congruent.

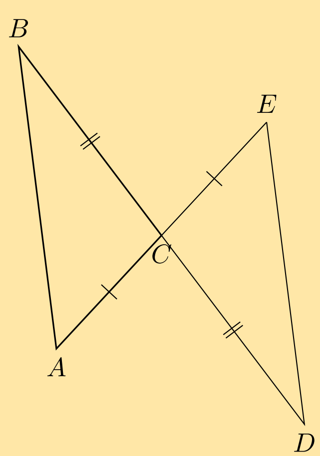
5. Have a look at the following triangles, which are drawn to scale:



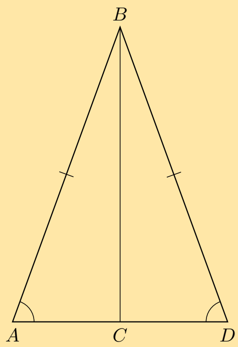
Are the two triangles congruent? If so state the reason and use the correct notation to state that they are congruent.

6. State whether the following pairs of triangles are congruent or not. Give reasons for your answers. If there is not enough information to make a decision, explain why.

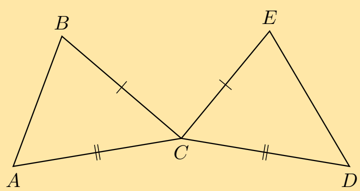
a)



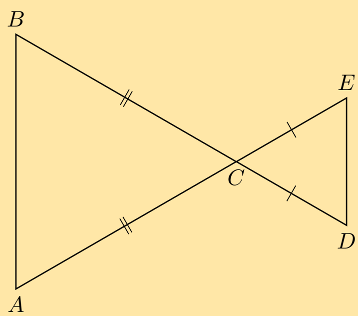
b)



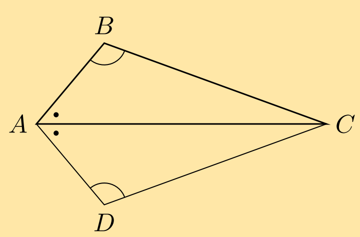
c)



d)



e)



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1a. 2G6G	1b. 2G6H	1c. 2G6J	1d. 2G6K	1e. 2G6M	1f. 2G6N	1g. 2G6P	2. 2G6Q
3. 2G6R	4. 2G6S	5. 2G6T	6a. 2G6V	6b. 2G6W	6c. 2G6X	6d. 2G6Y	6e. 2G6Z

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7.3 Quadrilaterals EMA5X

DEFINITION: *Quadrilateral*

A quadrilateral is a closed shape consisting of four straight line segments.

NOTE:
The interior angles of a quadrilateral add up to 360°.

Parallelogram EMA5Y

DEFINITION: *Parallelogram*

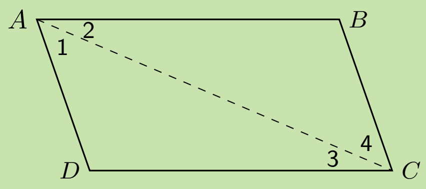
A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

Worked example 3: Properties of a parallelogram

QUESTION

$ABCD$ is a parallelogram with $AB \parallel DC$ and $AD \parallel BC$. Show that:

1. $AB = DC$ and $AD = BC$
2. $\hat{A} = \hat{C}$ and $\hat{B} = \hat{D}$



SOLUTION

Step 1: Connect AC to form $\triangle ABC$ and $\triangle CDA$
Redraw the diagram and draw line AC.

Step 2: Use properties of parallel lines to indicate all equal angles on the diagram
On your diagram mark all the equal angles.

Step 3: Prove $\triangle ABC \equiv \triangle CDA$
In $\triangle ABC$ and $\triangle CDA$:

$$\begin{aligned} \hat{A}_2 &= \hat{C}_3 && \text{(alt } \angle\text{s; } AB \parallel DC) \\ \hat{C}_4 &= \hat{A}_1 && \text{(alt } \angle\text{s; } BC \parallel AD) \\ AC & && \text{(common side)} \\ \therefore \triangle ABC &\equiv \triangle CDA && \text{(AAS)} \\ \therefore AB = CD & \quad \text{and} \quad BC = DA \end{aligned}$$

$\therefore$ Opposite sides of a parallelogram have equal length.

We have already shown $\hat{A}_2 = \hat{C}_3$ and $\hat{A}_1 = \hat{C}_4$. Therefore,

$$\hat{A} = \hat{A}_1 + \hat{A}_2 = \hat{C}_3 + \hat{C}_4 = \hat{C}$$

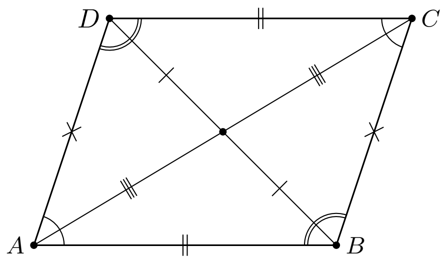
Furthermore,

$$\hat{B} = \hat{D} \quad (\triangle ABC \equiv \triangle CDA)$$

Therefore opposite angles of a parallelogram are equal.

Summary of the properties of a parallelogram:

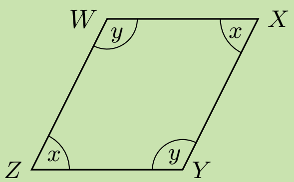
- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.



Worked example 4: Proving a quadrilateral is a parallelogram

QUESTION

Prove that if both pairs of opposite angles in a quadrilateral are equal, the quadrilateral is a parallelogram.



SOLUTION

Step 1: Find the relationship between $\hat{x}$ and $\hat{y}$

In $WXYZ$:

$$\begin{aligned} \hat{W} = \hat{Y} &= \hat{y} && \text{(given)} \\ \hat{Z} = \hat{X} &= \hat{x} && \text{(given)} \\ \hat{W} + \hat{X} + \hat{Y} + \hat{Z} &= 360^\circ && \text{(sum of } \angle\text{s in a quad)} \\ \therefore 2\hat{x} + 2\hat{y} &= 360^\circ \\ \therefore \hat{x} + \hat{y} &= 180^\circ \\ \hat{W} + \hat{Z} &= \hat{x} + \hat{y} \\ &= 180^\circ \end{aligned}$$

But these are co-interior angles between lines WX and ZY. Therefore $WX \parallel ZY$.

Step 2: Find parallel lines

Similarly $\hat{W} + \hat{X} = 180^\circ$. These are co-interior angles between lines XY and WZ. Therefore $XY \parallel WZ$.

Both pairs of opposite sides of the quadrilateral are parallel, therefore $WXYZ$ is a parallelogram.

Investigation: Proving a quadrilateral is a parallelogram

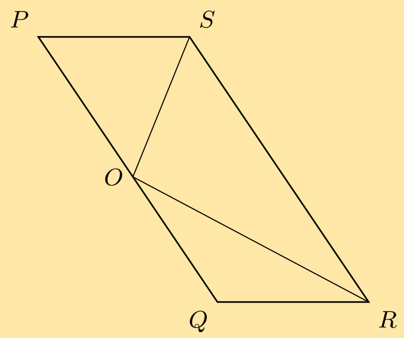
1. Prove that if both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
2. Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
3. Prove that if one pair of opposite sides of a quadrilateral are both equal and parallel, then the quadrilateral is a parallelogram.

A quadrilateral is a parallelogram if:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal.
- Both pairs of opposite angles are equal.
- The diagonals bisect each other.
- One pair of opposite sides are both equal and parallel.

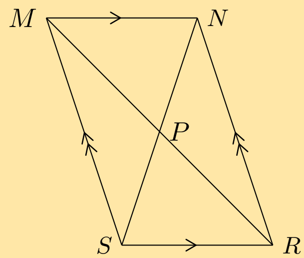
Exercise 7 – 3:

1. $PQRS$ is a parallelogram. $PS = OS$ and $QO = QR$. $S\hat{O}R = 96°$ and $Q\hat{O}R = x$.



   a) Find with reasons, two other angles equal to x.
   b) Write $\hat{P}$ in terms of x.
   c) Calculate the value of x.

2. Prove that the diagonals of parallelogram $MNRS$ bisect one another at P.



Hint: Use congruency.

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Rectangle EMA5Z

DEFINITION: *Rectangle*

A rectangle is a parallelogram that has all four angles equal to $90°$.

A rectangle has all the properties of a parallelogram:

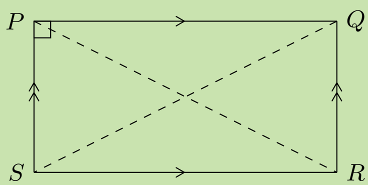
- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.

It also has the following special property:

Worked example 5: Special property of a rectangle

QUESTION

$PQRS$ is a rectangle. Prove that the diagonals are of equal length.



SOLUTION

Step 1: Connect P to R and Q to S to form $\triangle PSR$ and $\triangle QRS$

Step 2: Use the definition of a rectangle to fill in on the diagram all equal angles and sides

Step 3: Prove $\triangle PSR \equiv \triangle QRS$

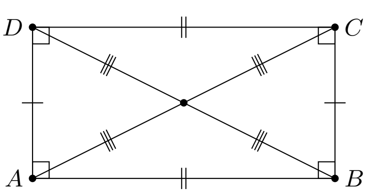
In $\triangle PSR$ and $\triangle QRS$:

$$\begin{aligned}
PS &= QR && \text{(opp sides of rectangle)} \\
SR & && amp; \text{(common side)} \\
P\hat{S}R &= Q\hat{R}S = 90° && (\angle\text{s of rectangle}) \\
\therefore \triangle PSR &\equiv \triangle QRS && \text{(RHS)} \\
\text{Therefore } PR &= QS
\end{aligned}$$

The diagonals of a rectangle are of equal length.

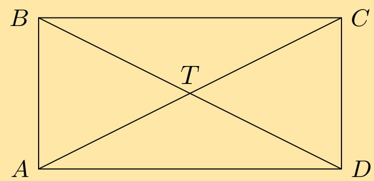
Summary of the properties of a rectangle:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are of equal length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- Diagonals are equal in length.
- All interior angles are equal to 90°



Exercise 7 – 4:

1. $ABCD$ is a quadrilateral. Diagonals AC and BD intersect at T. $AC = BD$, $AT = TC$, $DT = TB$.



   Prove that:

   a) $ABCD$ is a parallelogram
   b) $ABCD$ is a rectangle

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Rhombus EMA62

DEFINITION: *Rhombus*

A rhombus is a parallelogram with all four sides of equal length.

A rhombus has all the properties of a parallelogram:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.

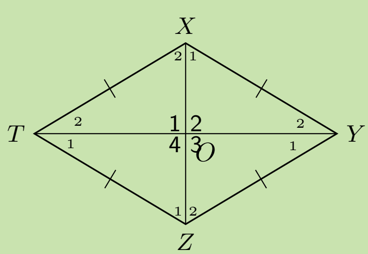
It also has two special properties:

Worked example 6: Special properties of a rhombus

QUESTION

$XYZT$ is a rhombus. Prove that:

1. the diagonals bisect each other perpendicularly;
2. the diagonals bisect the interior angles.



SOLUTION

Step 1: Use the definition of a rhombus to fill in on the diagram all equal angles and sides

Step 2: Prove $\triangle XTO \equiv \triangle ZTO$

$$
\begin{aligned}
XT &= ZT && \text{(sides of rhombus)} \\
TO &\ amp; && \text{(common side)} \\
XO &= OZ && \text{(diags of rhombus)} \\
\therefore \triangle XTO &\equiv \triangle ZTO && \text{(SSS)} \\
\therefore \hat{O}_1 &= \hat{O}_4 && \\
\text{But } \hat{O}_1 + \hat{O}_4 &= 180° && (\angle\text{s on a str line}) \\
\therefore \hat{O}_1 &= \hat{O}_4 = 90° &&
\end{aligned}
$$

We can further conclude that $\hat{O}_1 = \hat{O}_2 = \hat{O}_3 = \hat{O}_4 = 90°$.

Therefore the diagonals bisect each other perpendicularly.

Step 3: Use properties of congruent triangles to prove diagonals bisect interior angles

$$
\begin{aligned}
\hat{X}_2 &= \hat{Z}_1 && (\triangle XTO \equiv \triangle ZTO) \\
\text{and } \hat{X}_2 &= \hat{Z}_2 && (\text{alt } \angle\text{s; } XT \parallel YZ) \\
\therefore \hat{Z}_1 &= \hat{Z}_2 &&
\end{aligned}
$$

Therefore diagonal XZ bisects $\hat{Z}$. Similarly, we can show that XZ also bisects $\hat{X}$; and that diagonal TY bisects $\hat{T}$ and $\hat{Y}$.

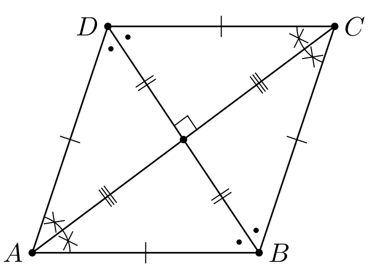
We conclude that the diagonals of a rhombus bisect the interior angles.

To prove a parallelogram is a rhombus, we need to show any one of the following:

- All sides are equal in length.
- Diagonals intersect at right angles.
- Diagonals bisect interior angles.

Summary of the properties of a rhombus:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- All sides are equal in length.
- The diagonals bisect each other at 90°
- The diagonals bisect both pairs of opposite angles.



Square

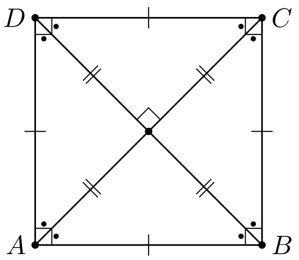
EMA63

DEFINITION: *Square*

A square is a rhombus with all four interior angles equal to $90°$
OR
A square is a rectangle with all four sides equal in length.

A square has all the properties of a rhombus:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- All sides are equal in length.
- The diagonals bisect each other at $90°$
- The diagonals bisect both pairs of opposite angles.



It also has the following special properties:

- All interior angles equal $90°$.
- Diagonals are equal in length.
- Diagonals bisect both pairs of interior opposite angles (i.e. all are $45°$).

To prove a parallelogram is a square, we need to show either one of the following:

- It is a rhombus (all four sides of equal length) with interior angles equal to $90°$.
- It is a rectangle (interior angles equal to $90°$).

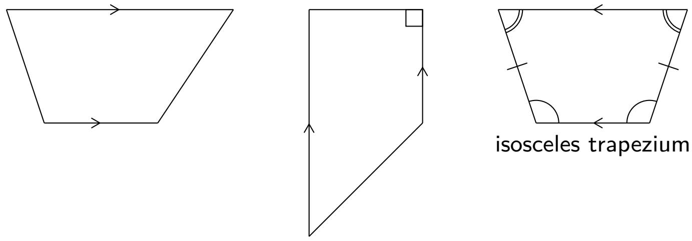
Trapezium EMA64

DEFINITION: *Trapezium*

A trapezium is a quadrilateral with one pair of opposite sides parallel.

NOTE:
A trapezium is sometimes called a trapezoid.

Some examples of trapeziums are given below:



Kite EMA65

DEFINITION: *Kite*

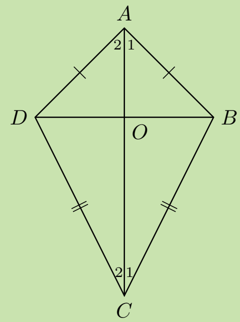
A kite is a quadrilateral with two pairs of adjacent sides equal.

Worked example 7: Properties of a kite

QUESTION

$ABCD$ is a kite with $AD = AB$ and $CD = CB$. Prove that:

1. $A\hat{D}C = A\hat{B}C$
2. Diagonal AC bisects $\hat{A}$ and $\hat{C}$



SOLUTION

Step 1: Prove $\triangle ADC \equiv \triangle ABC$

In $\triangle ADC$ and $\triangle ABC$:

$$\begin{aligned} AD &= AB && \text{(given)} \\ CD &= CB && \text{(given)} \\ AC & && \text{(common side)} \\ \therefore \triangle ADC &\equiv \triangle ABC && \text{(SSS)} \\ \therefore A\hat{D}C &= A\hat{B}C \end{aligned}$$

Therefore one pair of opposite angles are equal in kite $ABCD$.

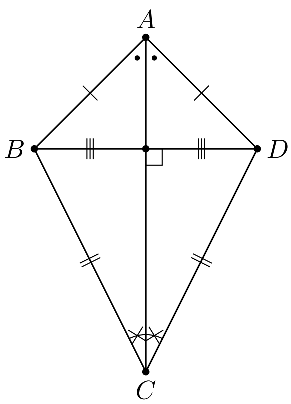
Step 2: Use properties of congruent triangles to prove AC bisects $\hat{A}$ and $\hat{C}$

$$\begin{aligned} \hat{A}_1 &= \hat{A}_2 && (\triangle ADC \equiv \triangle ABC) \\ \text{and } \hat{C}_1 &= \hat{C}_2 && (\triangle ADC \equiv \triangle ABC) \end{aligned}$$

Therefore diagonal AC bisects $\hat{A}$ and $\hat{C}$.

We conclude that the diagonal between the equal sides of a kite bisects the two interior angles and is an axis of symmetry.

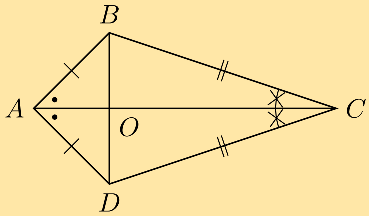
Summary of the properties of a kite:



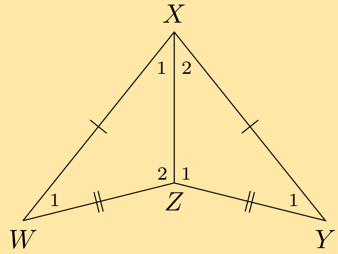
- Diagonal between equal sides bisects the other diagonal.
- One pair of opposite angles are equal (the angles between unequal sides).
- Diagonal between equal sides bisects the interior angles and is an axis of symmetry.
- Diagonals intersect at $90°$

Exercise 7 – 5:

1. Use the sketch of quadrilateral $ABCD$ to prove the diagonals of a kite are perpendicular to each other.



2. Explain why quadrilateral $WXYZ$ is a kite. Write down all the properties of quadrilateral $WXYZ$.



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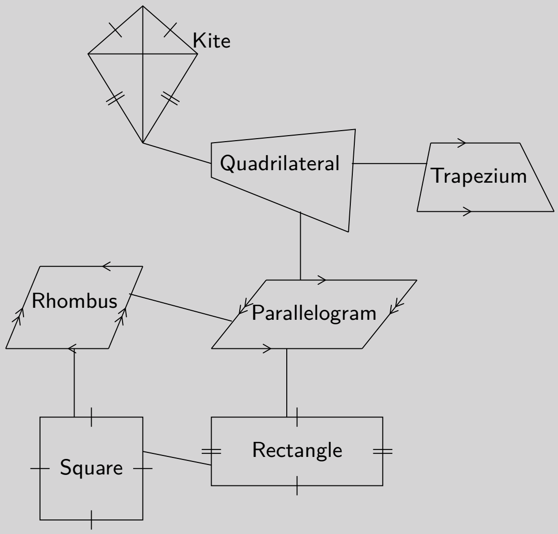
VISIT:
This video provides a summary of the different types of quadrilaterals and their properties.
See video: 2G77 at www.everythingmaths.co.za

Investigation: Relationships between the different quadrilaterals

Heather has drawn the following diagram to illustrate her understanding of the relationships between the different quadrilaterals. The following diagram summarises the different types of special quadrilaterals.

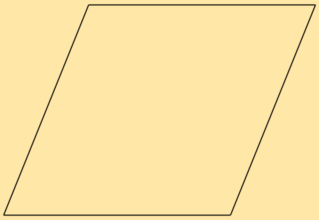


1. Explain her possible reasoning for structuring the diagram as shown.
2. Design your own diagram to show the relationships between the different quadrilaterals and write a short explanation of your design.

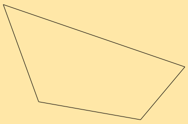
Exercise 7 – 6:

1. The following shape is drawn **to scale**:

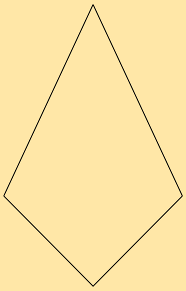
   Give the most specific name for the shape.
2. The following shape is drawn **to scale**:



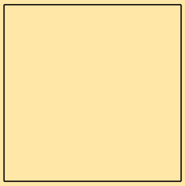
   Give the most specific name for the shape.
3. Based on the shape that you see list the all the names of the shape. The figure is drawn to scale.



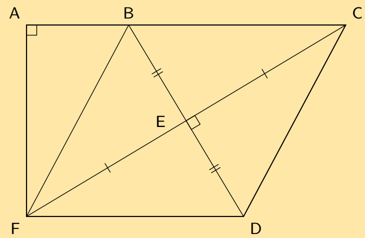
4. Based on the shape that you see list the all the names of the shape. The figure is drawn to scale.



5. Based on the shape that you see list the all the names of the shape. The figure is drawn to scale.



6. Find the area of $ACDF$ if $AB = 8,\ BF = 17,\ FE = EC,\ BE = ED,\ \hat{A} = 90°,\ C\hat{E}D = 90°$.



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1. 2G78 2. 2G79 3. 2G7B 4. 2G7C 5. 2G7D 6. 2G7F

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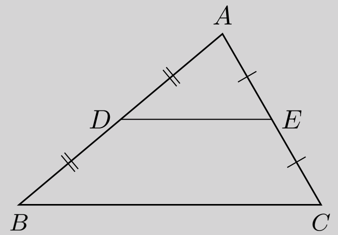
7.4 The mid-point theorem EMA66

Investigation: Proving the mid-point theorem

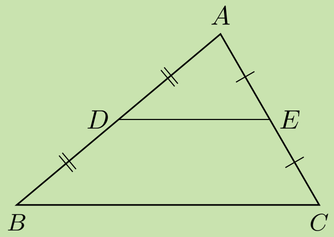


1. Draw a large scalene triangle on a sheet of paper.
2. Name the vertices A, B and C. Find the mid-points (D and E) of two sides and connect them.
3. Cut out $\triangle ABC$ and cut along line DE.
4. Place $\triangle ADE$ on quadrilateral $BDEC$ with vertex E on vertex C. Write down your observations.
5. Shift $\triangle ADE$ to place vertex D on vertex B. Write down your observations.
6. What do you notice about the lengths DE and BC?
7. Make a conjecture regarding the line joining the mid-point of two sides of a triangle.

Worked example 8: Mid-point theorem

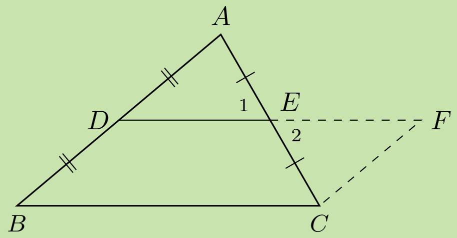
QUESTION

Prove that the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.



SOLUTION

Step 1: Extend DE to F so that $DE = EF$ and join FC



Step 2: Prove $BCFD$ is a parallelogram

In $\triangle EAD$ and $\triangle ECF$:

$$\begin{aligned} \hat{E}_1 &= \hat{E}_2 && \text{(vert opp } \angle\text{s} = \text{)} \\ AE &= CE && \text{(given)} \\ DE &= EF && \text{(by construction)} \\ \therefore \triangle EAD &\equiv \triangle ECF && \text{(SAS)} \\ \therefore A\hat{D}E &= C\hat{F}E \end{aligned}$$

But these are alternate interior angles, therefore $BD \parallel FC$

$$\begin{aligned} BD &= DA && \text{(given)} \\ DA &= FC && (\triangle EAD \equiv \triangle ECF) \\ \therefore BD &= FC \\ \therefore BCFD & \text{ is a parallelogram} && \text{(one pair opp. sides } = \text{ and } \parallel \text{)} \end{aligned}$$

Therefore $DE \parallel BC$.

We conclude that the line joining the two mid-points of two sides of a triangle is parallel to the third side.

Step 3: Use properties of parallelogram $BCFD$ to prove that $DE = \frac{1}{2}BC$

$$\begin{aligned} DF &= BC && \text{(opp sides } \parallel \text{m)} \\ \text{and } DF &= 2\,(DE) && \text{(by construction)} \\ \therefore 2DE &= BC \\ \therefore DE &= \tfrac{1}{2}BC \end{aligned}$$

We conclude that the line joining the mid-point of two sides of a triangle is equal to half the length of the third side.

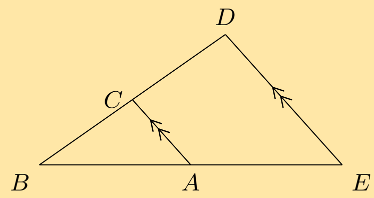
Converse

The converse of this theorem states: If a line is drawn through the mid-point of a side of a triangle parallel to the second side, it will bisect the third side.

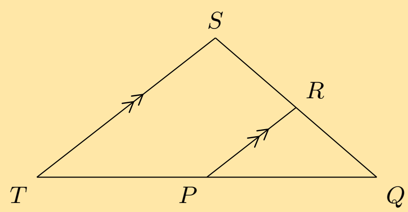
VISIT:
You can use GeoGebra to show that the converse of the mid-point theorem is true.

Exercise 7 – 7:

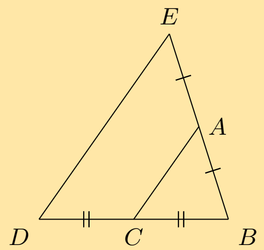
1. Points C and A are the mid-points on lines BD and BE. Study $\triangle EDB$ carefully. Identify the third side of this triangle, using the information as shown, together with what you know about the mid-point theorem. Name the third side by its endpoints, e.g., FG.



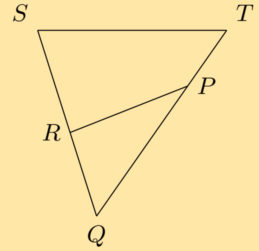
2. Points R and P are the mid-points on lines QS and QT. Study $\triangle TSQ$ carefully. Identify the third side of this triangle, using the information as shown, together with what you know about the mid-point theorem. Name the third side by its endpoints, e.g., FG.



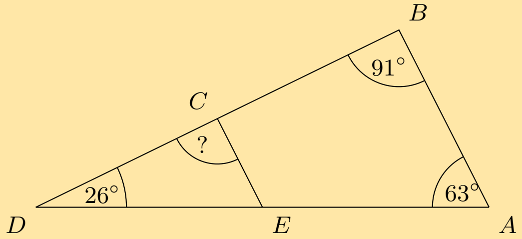
3. Points C and A are given on the lines BD and BE. Study the triangle carefully, then identify and name the parallel lines.



4. Points R and P are given on the lines QS and QT. Study the triangle carefully, then identify and name the parallel lines.

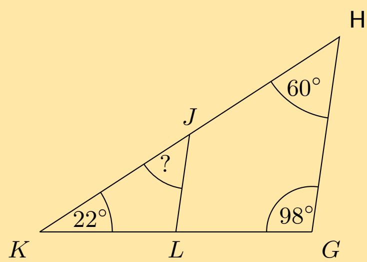


5. The figure below shows a large triangle with vertices A, B and D, and a smaller triangle with vertices at C, D and E. Point C is the mid-point of BD and point E is the mid-point of AD.



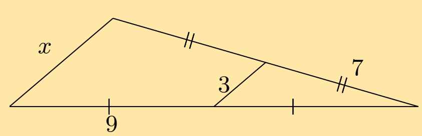
   a) Three angles are given: $\hat{A} = 63°$, $\hat{B} = 91°$ and $\hat{D} = 26°$; determine the value of $D\hat{C}E$.
   b) The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).
   $\triangle DEC \ ||| \ \triangle ?$

6. The figure below shows a large triangle with vertices G, H and K, and a smaller triangle with vertices at J, K and L. Point J is the mid-point of HK and point L is the mid-point of GK.

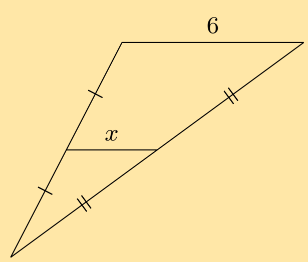


   a) Three angles are given: $\hat{G} = 98°$, $\hat{H} = 60°$, and $\hat{K} = 22°$; determine the value of $K\hat{J}L$.
   b) The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).
   $\triangle HKG \ ||| \ \triangle ?$

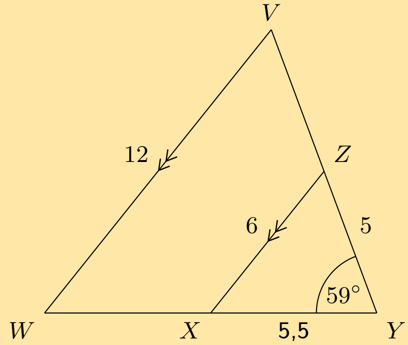
7. Consider the triangle in the diagram below. There is a line crossing through a large triangle. Notice that some lines in the figure are marked as equal to each other. One side of the triangle has a given length of 3. Determine the value of x.



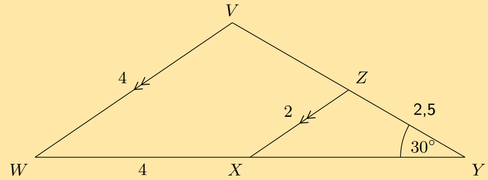
8. Consider the triangle in the diagram below. There is a line crossing through a large triangle. Notice that some lines in the figure are marked as equal to each other. One side of the triangle has a given length of 6. Determine the value of x.



9. In the figure below, $VW \parallel ZX$, as labelled. Furthermore, the following lengths and angles are given: $VW = 12$; $ZX = 6$; $XY = 5{,}5$; $YZ = 5$ and $\hat{V} = 59°$. The figure is drawn to scale. Determine the length of WY.

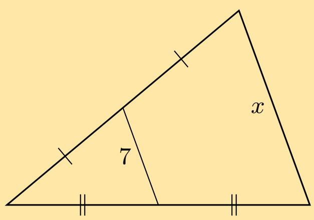


10. In the figure below, $VW \parallel ZX$, as labelled. Furthermore, the following lengths and angles are given: $VW = 4$; $ZX = 2$; $WX = 4$; $YZ = 3{,}5$ and $\hat{Y} = 30°$. The figure is drawn to scale. Determine the length of XY.

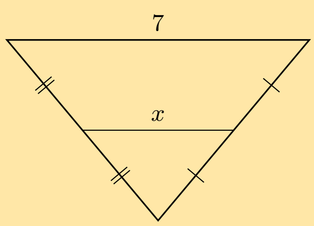


11. Find x and y in the following:

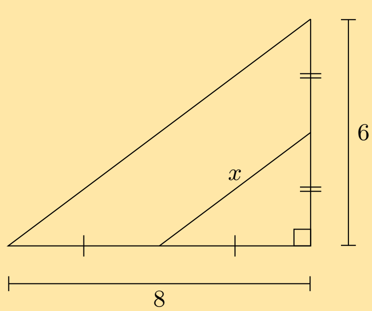
a)



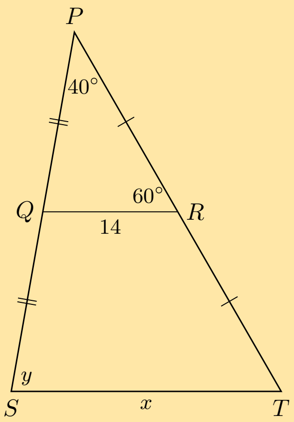
b)



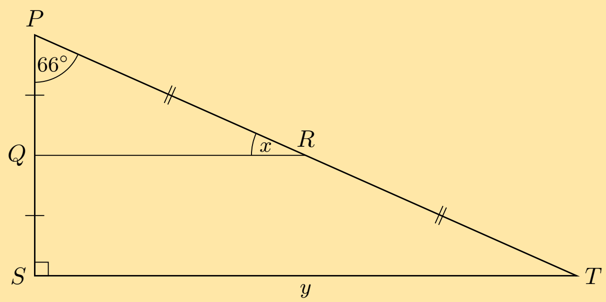
c)



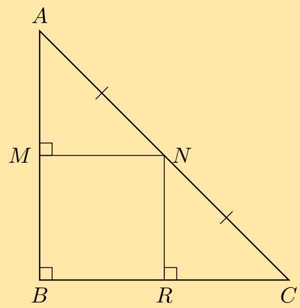
d)



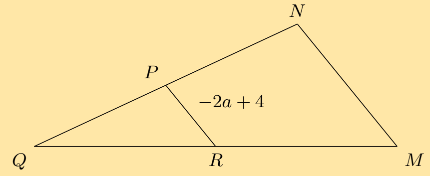
e) In the following diagram $PQ = 2{,}5$ and $RT = 6{,}5$.



12. Show that M is the mid-point of AB and that $MN = RC$.



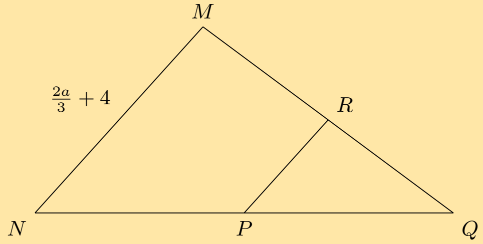
13. In the diagram below, P is the mid-point of NQ and R is the mid-point of MQ. The segment inside of the large triangle is labelled with a length of $-2a + 4$.



a) Calculate the value of MN in terms of a.

b) You are now told that MN has a length of 18. What is the value of a? Give your answer as a fraction.

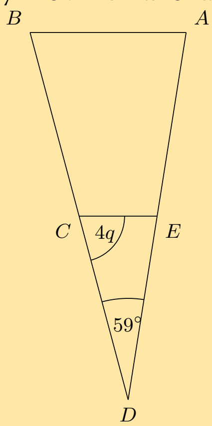
14. In the diagram below, P is the mid-point of NQ and R is the mid-point of MQ. One side of the triangle has a given length of $\frac{2a}{3} + 4$.



a) Find the value of PR in terms of a.

b) You are now told that PR has a length of 8. What is the value of a?

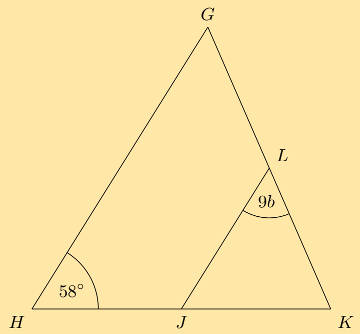
15. The figure below shows $\triangle ABD$ crossed by EC. Points C and E bisect their respective sides of the triangle.



a) The angles $\hat{D} = 59°$ and $E\hat{C}D = 4q$ are given; determine the value of $\hat{A}$ in terms of q.

b) You are now told that $E\hat{C}D$ has a measure of $72°$. Calculate for the value of q.

16. The figure below shows $\triangle GHK$ crossed by LJ. Points J and L bisect their respective sides of the triangle.



a) Given the angles $\hat{H} = 58°$ and $K\hat{L}J = 9b$, determine the value of $\hat{K}$ in terms of b.

b) You are now told that $\hat{K}$ has a measure of $74°$. Solve for the value of b. Give your answer as a fraction.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2G7G 2. 2G7H 3. 2G7J 4. 2G7K 5. 2G7M 6. 2G7N 7. 2G7P 8. 2G7Q
9. 2G7R 10. 2G7S 11a. 2G7T 11b. 2G7V 11c. 2G7W 11d. 2G7X 11e. 2G7Y 12. 2G7Z
13. 2G82 14. 2G83 15. 2G84 16. 2G85

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7.5 Chapter summary

EMA67

See presentation: 2G86 at www.everythingmaths.co.za

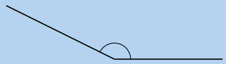
- A quadrilateral is a closed shape consisting of four straight line segments.
- A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
  - Both pairs of opposite sides are equal in length.
  - Both pairs of opposite angles are equal.
  - Both diagonals bisect each other.
- A rectangle is a parallelogram that has all four angles equal to $90°$
  - Both pairs of opposite sides are parallel.
  - Both pairs of opposite sides are equal in length.
  - The diagonals bisect each other.
  - The diagonals are equal in length.
  - All interior angles are equal to $90°$.
- A rhombus is a parallelogram that has all four sides equal in length.
  - Both pairs of opposite sides are parallel.
  - All sides are equal in length.
  - Both pairs of opposite angles are equal.
  - The diagonals bisect each other at $90°$.
  - The diagonals of a rhombus bisect both pairs of opposite angles.
- A square is a rhombus that has all four interior angles equal to $90°$.
  - Both pairs of opposite sides are parallel.
  - The diagonals bisect each other at $90°$.
  - All interior angles are equal to $90°$.
  - The diagonals are equal in length.
  - The diagonals bisect both pairs of interior opposite angles (i.e. all are $45°$
- A trapezium is a quadrilateral with one pair of opposite sides parallel.
- A kite is a quadrilateral with two pairs of adjacent sides equal.
  - One pair of opposite angles are equal (the angles are between unequal sides).
  - The diagonal between equal sides bisects the other diagonal.
  - The diagonal between equal sides bisects the interior angles.
  - The diagonals intersect at $90°$.
- The mid-point theorem states that the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.

End of chapter Exercise 7 – 8:

1. Identify the types of angles shown below:

   a)

b)



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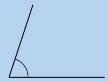

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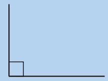
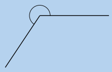
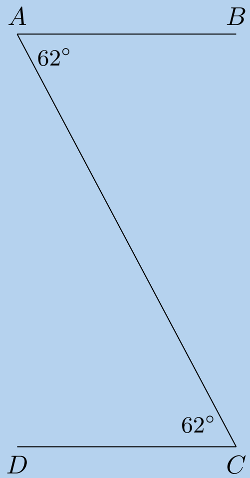


e)

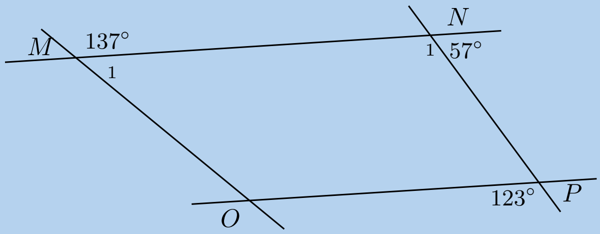


f) An angle of 91°
g) An angle of 180°
h) An angle of 210°

2. Assess whether the following statements are true or false. If the statement is false, explain why:
   a) A trapezium is a quadrilateral with two pairs of opposite sides that are parallel.
   b) Both diagonals of a parallelogram bisect each other.
   c) A rectangle is a parallelogram that has all interior angles equal to 90°.
   d) Two adjacent sides of a rhombus have different lengths.
   e) The diagonals of a kite intersect at right angles.
   f) All squares are parallelograms.
   g) A rhombus is a kite with a pair of equal, opposite sides.
   h) The diagonals of a parallelogram are axes of symmetry.
   i) The diagonals of a rhombus are equal in length.
   j) Both diagonals of a kite bisect the interior angles.
3. Find all pairs of parallel lines in the following figures, giving reasons in each case.
   a)



b)



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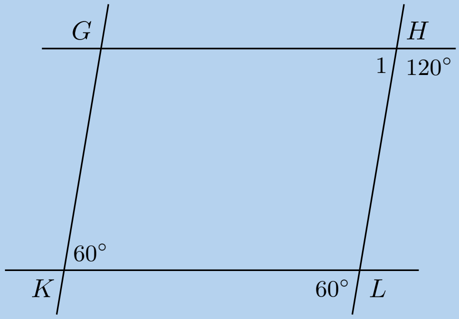
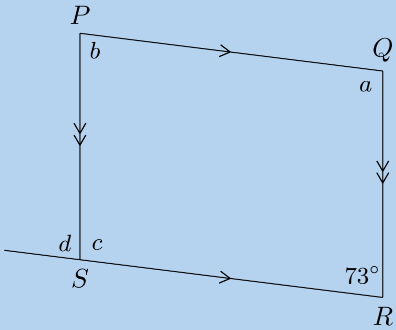
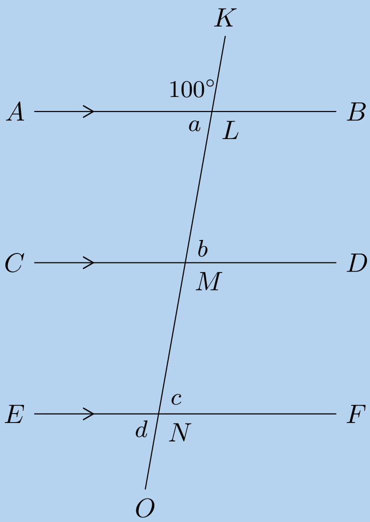


4. Find angles a, b, c and d in each case, giving reasons:

a)



b)



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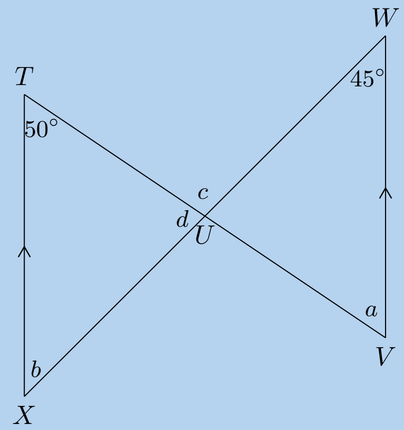
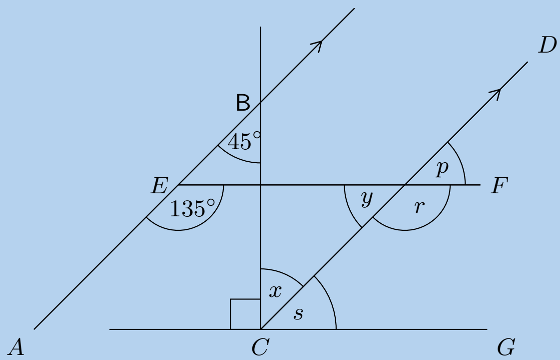


5. Given the figure below.



a) Find each of the unknown angles marked in the figure below. Find a reason that leads to the answer in a single step.
b) Based on the results for the angles above, is $EF \parallel CG$?

6. Given the following diagrams:

Diagram A

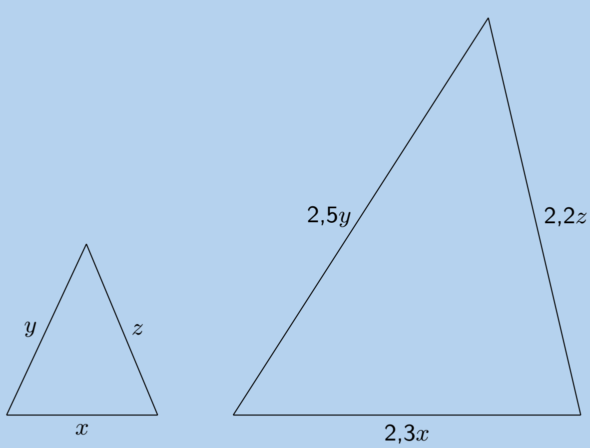
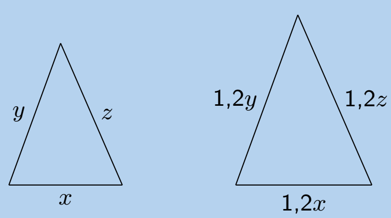


Diagram B



Which diagram correctly gives a pair of similar triangles?

7. Given the following diagrams:

Diagram A

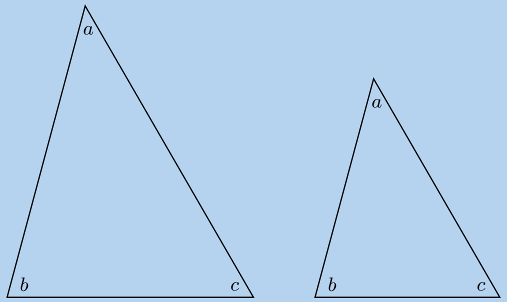
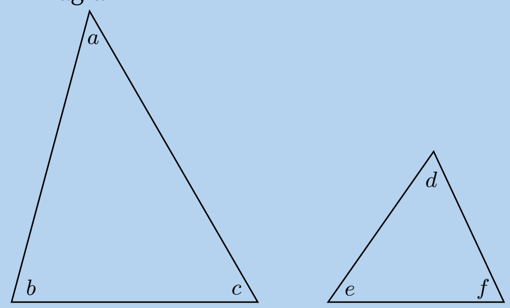
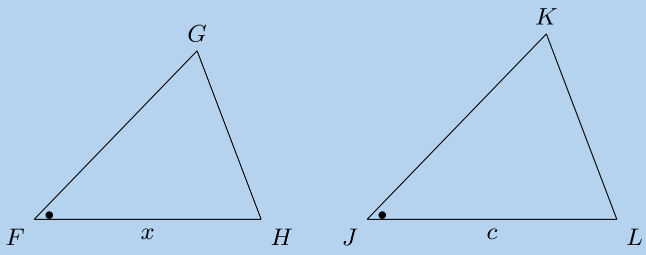


Diagram B



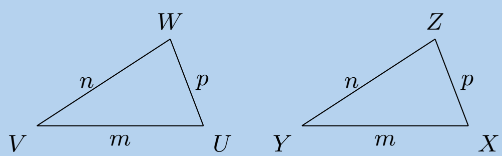
Which diagram correctly gives a pair of similar triangles?

8. Have a look at the following triangles, which are drawn to scale:



Are the triangles congruent? If so state the reason and use correct notation to state that they are congruent.

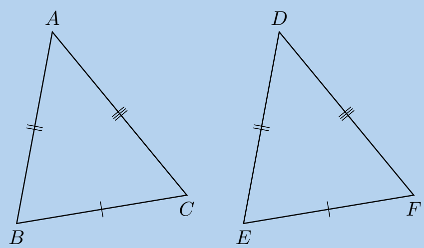
9. Have a look at the following triangles, which are drawn to scale:



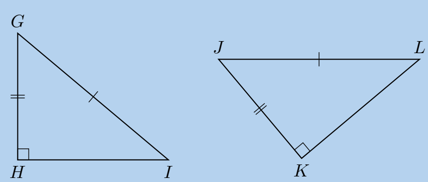
Are the triangles congruent? If so state the reason and use correct notation to state that they are congruent.

10. Say which of the following pairs of triangles are congruent with reasons.

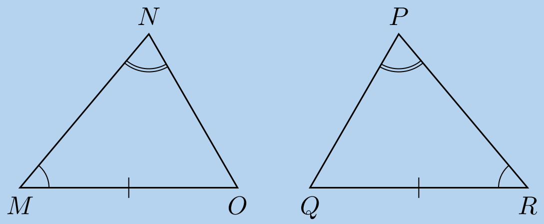
a)



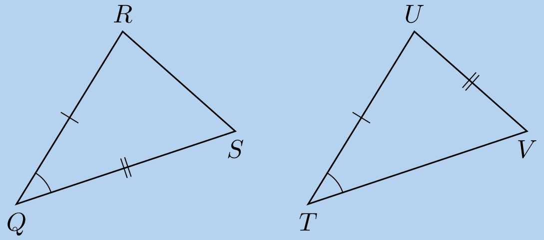
b)



c)

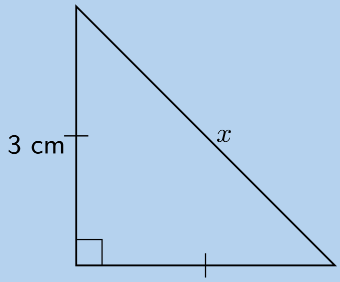


d)

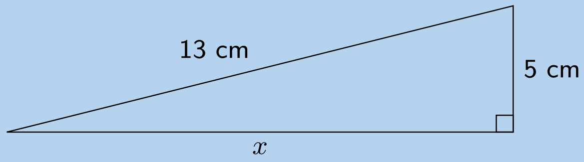


11. Using the theorem of Pythagoras, calculate the length x:

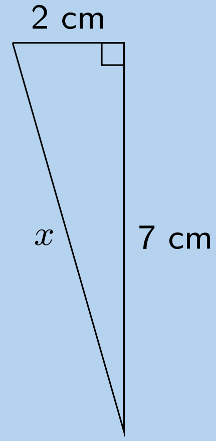
a)



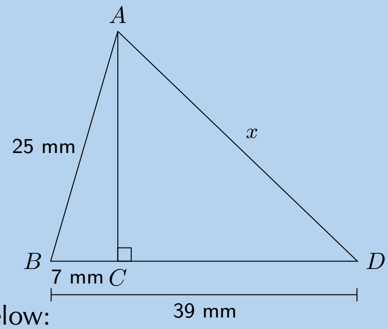
b)



c)

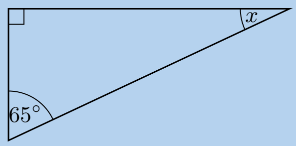


d)

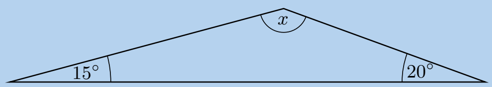


12. Calculate x and y in the diagrams below:

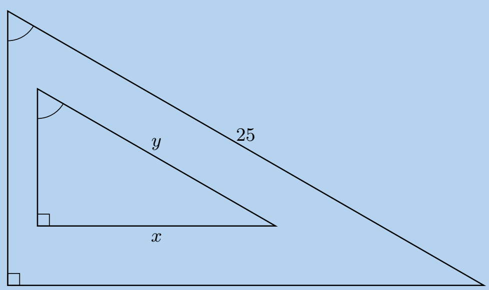
a)



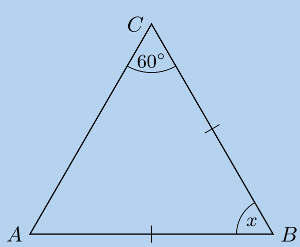
b)



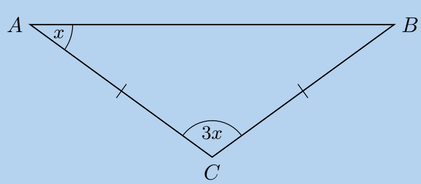
c)



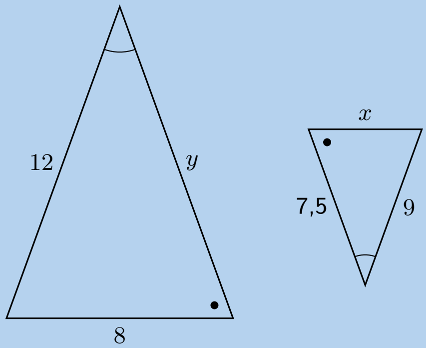
d)



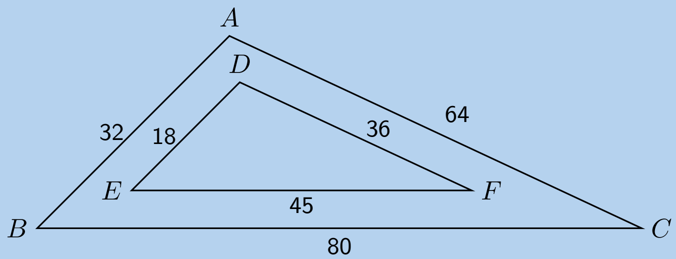
e)



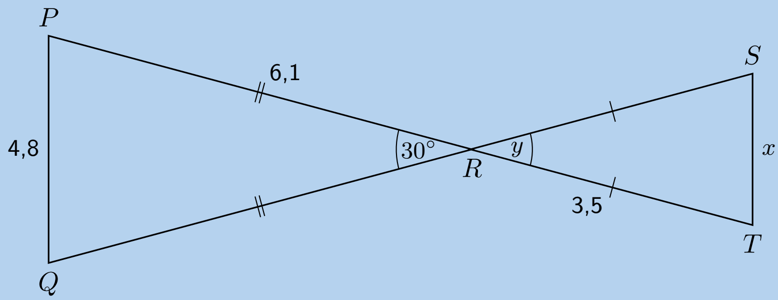
f)



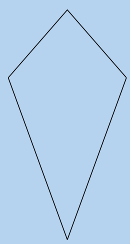
13. Consider the diagram below. Is $\triangle ABC \ ||| \ \triangle DEF$? Give reasons for your answer.



14. Explain why $\triangle PQR$ is similar to $\triangle TSR$ and calculate the values of x and y.

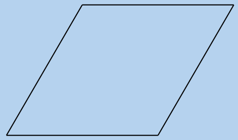


15. The following shape is drawn to scale:

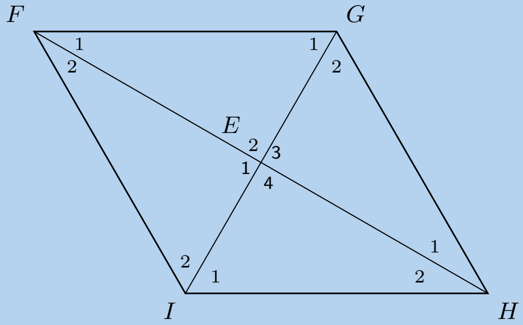


Give the most specific name for the shape.

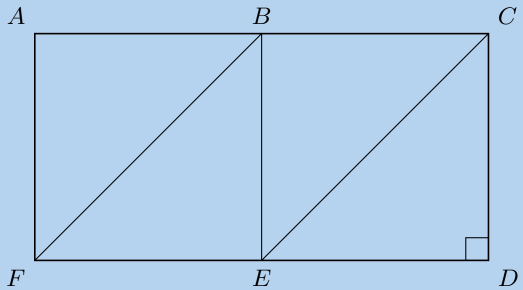
16. Based on the shape that you see list the all the names of the shape. The figure is drawn to scale.



17. $FGHI$ is a rhombus. $\hat{F}_1 = 3x + 20°$; $\hat{G}_1 = x + 10°$. Determine the value of x.



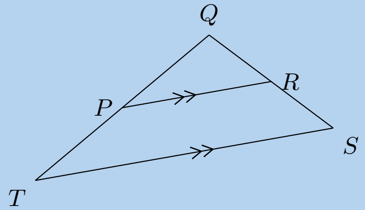
18. In the diagram below, $AB = BC = CD = DE = EF = FA = BE$.



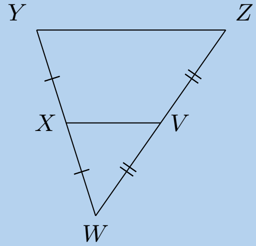
Name:

a) 3 rectangles
b) 4 parallelograms
c) 2 trapeziums
d) 2 rhombi

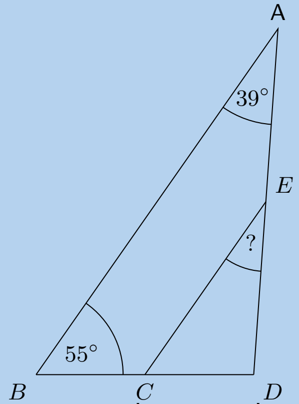
19. Points R and P are the mid-points on lines QS and QT. Study $\triangle TSQ$ carefully. Identify the third side of this triangle, using the information as shown, together with what you know about the mid-point theorem. (Name the third side by its endpoints, e.g., FG.)



20. Points X and V are given on the segments WY and WZ. Study the triangle carefully, then identify and name the parallel line segments.



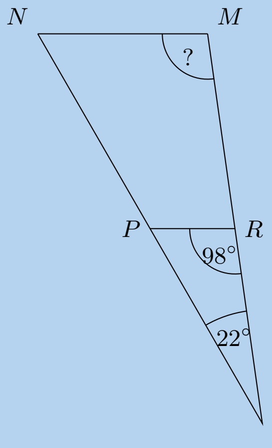
21. The figure below shows a large triangle with vertices A, B and D, and a smaller triangle with vertices at C, D and E. Point C is the mid-point of BD and point E is the mid-point of AD.



a) The angles $\hat{A} = 39°$ and $\hat{B} = 55°$ are given; determine the value of $D\hat{E}C$.

b) The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).
$\triangle DEC \;|||\; \triangle ?$

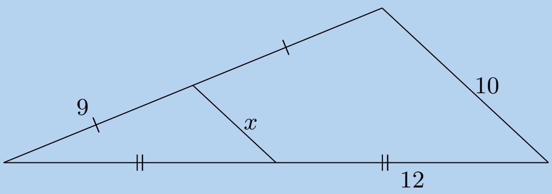
22. The figure below shows a large triangle with vertices M, N and Q, and a smaller triangle with vertices at P, Q and R. Point P is the mid-point of NQ and point R is the mid-point of MQ.



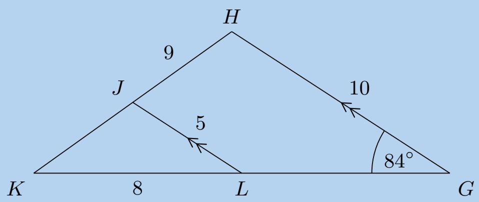
a) With the two angles given, $\hat{Q} = 22°$ and $Q\hat{R}P = 98°$, determine the value of $\hat{M}$.

b) The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).
$\triangle QMN \;|||\; \triangle ?$

23. Consider the triangle in the diagram below. There is a line segment crossing through a large triangle. Notice that some segments in the figure are marked as equal to each other. One side of the triangle has a given length of 10. Determine the value of x.

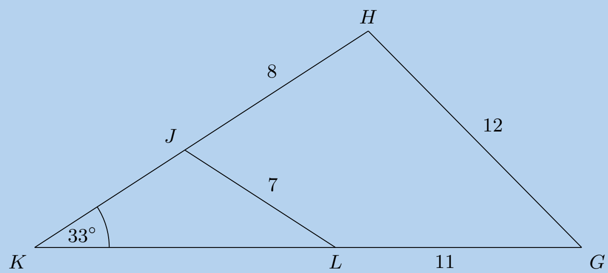


24. In the figure below, $GH \parallel LJ$, as labelled. Furthermore, the following lengths and angles are given: $GH = 10$; $LJ = 5$; $HJ = 9$; $KL = 8$ and $\hat{G} = 84°$. The figure is drawn to scale.



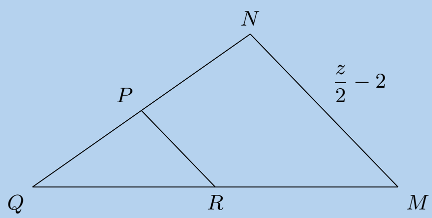
Calculate the length of JK.

25. The figure below shows triangle GHK with the smaller triangle JKL sitting inside of it. Furthermore, the following lengths and angles are given: $GH = 12; LJ = 7; HJ = 8; LG = 11$; $\hat{K} = 33°$. The figure is drawn to scale.



Find the length of KL.

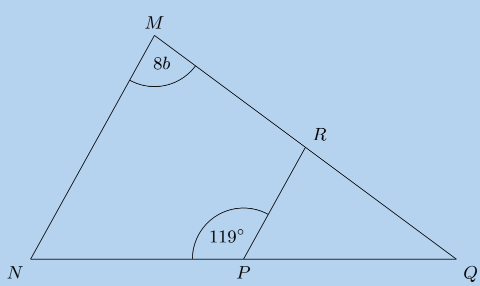
26. In the diagram below, P is the mid-point of NQ and R is the mid-point of MQ. One side of the triangle has a given length of $\frac{z}{2} - 2$.



a) Determine the value of PR in terms of z.

b) You are now told that PR has a length of 2. What is the value of z?

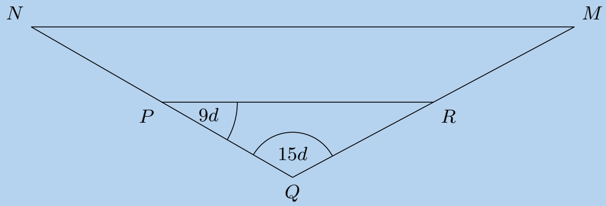
27. The figure below shows $\triangle MNQ$ crossed by RP. Points P and R bisect their respective sides of the triangle.



a) With the two angles given, $\hat{M} = 8b$ and $\angle NPR = 119°$, determine the value of $\hat{Q}$ in terms of b.

b) You are now told that $\hat{M}$ has a measure of $76°$. Determine for the value of b. Give your answer as an exact fractional value.

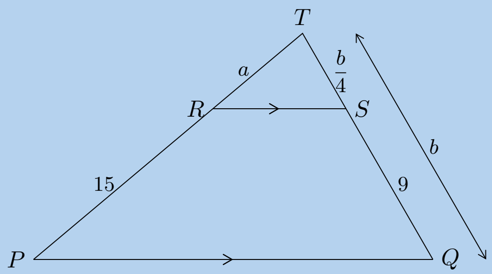
28. The figure below shows $\triangle MNQ$ crossed by RP. Points P and R bisect their respective sides of the triangle.



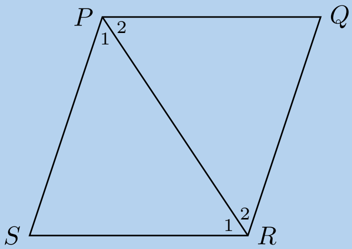
a) The angles $\hat{Q} = 15d$ and $R\hat{P}Q = 9d$ are given in the large triangle; determine the value of $\hat{M}$ in terms of d.

b) You are now told that $R\hat{P}Q$ has a measure of $60°$. Solve for the value of d. Give your answer as an exact fractional value.

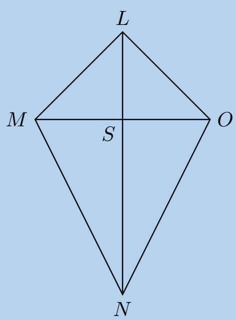
29. Calculate a and b:



30. $\triangle PQR$ and $\triangle PSR$ are equilateral triangles. Prove that $PQRS$ is a rhombus.



31. $LMNO$ is a quadrilateral with $LM = LO$ and diagonals that intersect at S such that $MS = SO$. Prove that:

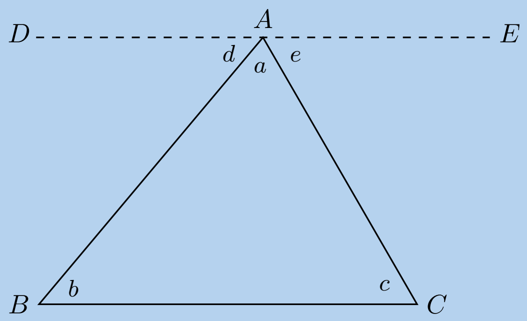


a) $M\hat{L}S = S\hat{L}O$

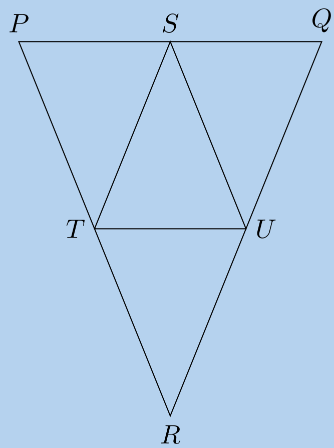
b) $\triangle LON \equiv \triangle LMN$

c) $MO \perp LN$

32. Using the figure below, show that the sum of the three angles in a triangle is 180°. Line DE is parallel to BC.



33. PQR is an isosceles triangle with $PR = QR$. S is the mid-point of PQ, T is the mid-point of PR and U is the mid-point of RQ.

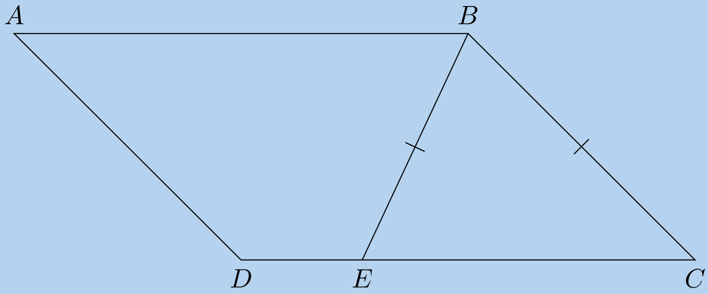


a) Prove $\triangle STU$ is also isosceles.

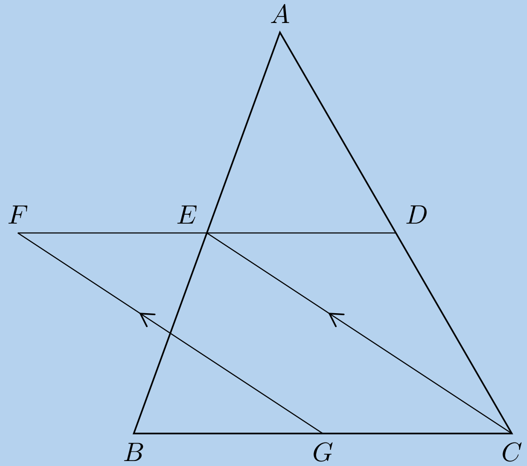
b) What type of quadrilateral is $STRU$? Motivate your answer.

c) If $R\hat{T}U = 68°$ calculate, with reasons, the size of $T\hat{S}U$.

34. $ABCD$ is a parallelogram. $BE = BC$. Prove that $A\hat{B}E = B\hat{C}D$.



35. In the diagram below, D, E and G are the mid-points of AC, AB and BC respectively. $EC \parallel FG$.



a) Prove that $FECG$ is a parallelogram.

b) Prove that $FE = ED$.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1a. 2G87	1b. 2G88	1c. 2G89	1d. 2G8B	1e. 2G8C	1f. 2G8D
1g. 2G8F	1h. 2G8G	2a. 2G8H	2b. 2G8J	2c. 2G8K	2d. 2G8M
2e. 2G8N	2f. 2G8P	2g. 2G8Q	2h. 2G8R	2i. 2G8S	2j. 2G8T
3a. 2G8V	3b. 2G8W	3c. 2G8X	4a. 2G8Y	4b. 2G8Z	4c. 2G92
5. 2G93	6. 2G94	7. 2G95	8. 2G96	9. 2G97	10a. 2G98
10b. 2G99	10c. 2G9B	10d. 2G9C	11a. 2G9D	11b. 2G9F	11c. 2G9G
11d. 2G9H	12a. 2G9J	12b. 2G9K	12c. 2G9M	12d. 2G9N	12e. 2G9P
12f. 2G9Q	13. 2G9R	14. 2G9S	15. 2G9T	16. 2G9V	17. 2G9W
18. 2G9X	19. 2G9Y	20. 2G9Z	21. 2GB2	22. 2GB3	23. 2GB4
24. 2GB5	25. 2GB6	26. 2GB7	27. 2GB8	28. 2GB9	29. 2GBB
30. 2GBC	31. 2GBD	32. 2GBF	33. 2GBG	34. 2GBH	35. 2GBJ

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