CHAPTER 14

Probability

We use probability to describe uncertain events. When you accidentally drop a slice of bread, you don't know if it's going to fall with the buttered side facing upwards or downwards. When your favourite sports team plays a game, you don't know whether they will win or not. When the weatherman says that there is a 40% chance of rain tomorrow, you may or may not end up getting wet. Uncertainty presents itself to some degree in every event that occurs around us and in every decision that we make.

We will see in this chapter that all of these uncertainties can be described using the rules of probability theory and that we can make definite conclusions about uncertain processes.

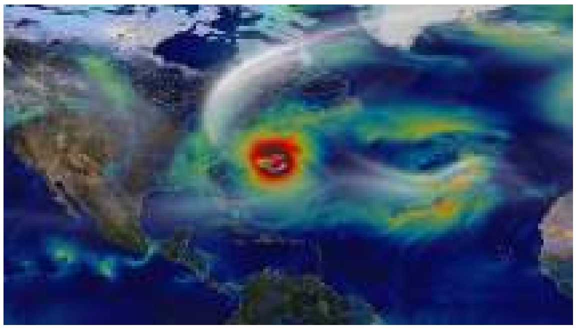


Figure 14.1: Tracking a superstorm. Meteorologists use computer software to help them track storms and predict the weather.

We'll use three examples of uncertain processes to help you understand the meanings of the different words used in probability theory: tossing a coin, rolling dice, and a soccer match.

VISIT:
The following video introduces the concepts used in probability.
See video: 2GVW at www.everythingmaths.co.za

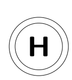
DEFINITION: *Experiment*

An experiment refers to an uncertain process.

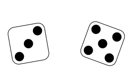
DEFINITION: *Outcome*

An outcome of an experiment is a single result of that experiment.

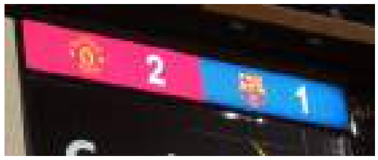
Experiment 1: A coin is tossed and it lands with either heads (H) or tails (T) facing upwards. An example outcome of tossing a coin is that it lands with heads facing up:



Experiment 2: Two dice are rolled and the total number of dots added up. An example outcome of rolling two dice:



Experiment 3: Two teams play a soccer match and we are interested in the final score. An example outcome of a soccer match:

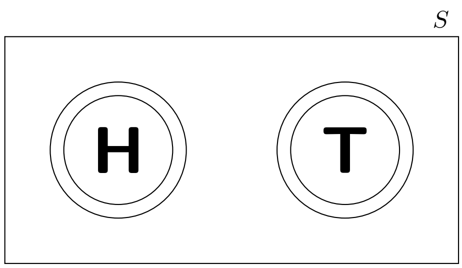


DEFINITION: *Sample space*

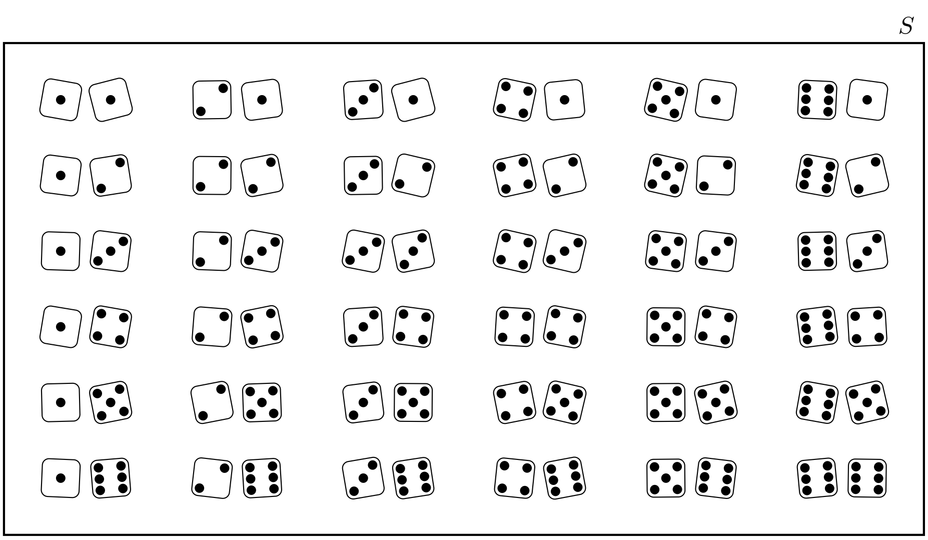
The sample space of an experiment is the set of all possible outcomes of that experiment. The sample space is denoted with the symbol S and the size of the sample space (the total number of possible outcomes) is denoted with $n\left(S\right)$

Even though we are usually interested in the outcome of an experiment, we also need to know what the other outcomes could have been. Let's have a look at the sample spaces of each of our three experiments.

Experiment 1: Since a coin can land in one of only two ways (we will ignore the possibility that the coin lands on its edge), the sample space is the set $S = \{\text{H}; \text{T}\}$. The size of the sample space of the coin toss is $n\left(S\right) = 2$:



Experiment 2: Each of the dice can land on a number from 1 to 6. In this experiment the sample space of all possible outcomes is every possible combination of the 6 numbers on the first die with the 6 numbers on the second die. This gives a total of $n\left(S\right) = 6 \times 6 = 36$ possible outcomes. The figure below shows all of the outcomes in the sample space of rolling two dice:



Experiment 3: Each soccer team can get an integer score from 0 upwards. Usually we don't expect a score to go much higher than 5 goals, but there is no reason why this cannot happen. So the sample space of this experiment consists of all possible combinations of two non-negative integers. The figure below shows all of the possibilities. Since we do not limit the score of a team, this sample space is infinitely large:

S

0 – 0	1 – 0	2 – 0	3 – 0	$\cdots$
0 – 1	1 – 1	2 – 1	3 – 1	$\cdots$
0 – 2	1 – 2	2 – 2	3 – 2	$\cdots$
0 – 3	1 – 3	2 – 3	3 – 3	
$\vdots$	$\vdots$	$\vdots$		$\ddots$

NOTE:
When we represent a sample space containing real numbers we can either write out all the outcomes in the sample space: $\{1;2;3;4;5;6;7;8;9;10\}$ or we can represent the sample space as: $\{n : n \epsilon \mathbb{Z},\ 1 \leq n \leq 10\}$.

DEFINITION: *Event*

An event is a specific set of outcomes of an experiment that you are interested in. An event is denoted with the letter E and the number of outcomes in the event with $n(E)$.

Experiment 1: Let us say that we would like the coin to land heads up. Here the event contains a single outcome: $E = \{\text{H}\}$. The size of the event set is $n(E) = 1$.

Experiment 2: Let us say that we are interested in the sum of the dice being 8. In this case the event set is:

E = {(⚁ ⚅); (⚂ ⚄); (⚃ ⚃); (⚄ ⚂); (⚅ ⚁)}

since it contains all of the possible ways to get 8 dots with 2 dice. The size of the event set is $n(E) = 5$.

Experiment 3: We would like to know whether the first team will win. For this event to happen the first score must be greater than the second.

$E = \{(1;0)\,;(2;0)\,;(2;1)\,;(3;0)\,;(3;1)\,;(3;2)\,;...\}$. This event set is infinitely large.

14.1 Theoretical probability EMA7W

DEFINITION: *Probability*

A probability is a real number between 0 and 1 that describes how likely it is that an event will occur.

We can describe probabilities in three ways:

1. As a real number between 0 and 1. For example 0,75.
2. As a percentage. For example 0,75 can be written as 75%.
3. As a fraction. For example 0,75 can also be written as $\frac{3}{4}$.

We note the following about probabilities:

- A probability of 0 means that an event will never occur.
- A probability of 1 means that an event will always occur.
- A probability of 0,5 means that an event will occur half the time, or 1 time out of every 2.

When all of the possible outcomes of an experiment have an equal chance of occurring, we can compute the exact theoretical probability of an event. The probability of an event is the ratio between the number of outcomes in the event set and the number of possible outcomes in the sample space.

$$P(E) = \frac{n(E)}{n(S)}$$

VISIT:
The following video shows an example of calculating the theoretical probabilities of an event.
See video: 2GVX at www.everythingmaths.co.za

Worked example 1: Theoretical probabilities

QUESTION

What is the theoretical probability of each of the events in the first two of our three experiments?

SOLUTION

Step 1: Write down the value of $n(S)$

Experiment 1 (coin): $n(S) = 2$

Experiment 2 (dice): $n(S) = 36$

Step 2: Write down the size of the event set

Experiment 1: $n(E) = 1$

Experiment 2: $n(E) = 5$

Step 3: Compute the theoretical probability

Experiment 1:

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{2} = 0{,}5$$

Experiment 2:

$$P(E) = \frac{n(E)}{n(S)} = \frac{5}{36} = 0{,}13\dot{8}$$

Note that we do not consider the theoretical probability of the third experiment. The third experiment is different from the first two in an important way, namely that all possible outcomes (all final scores) are not

equally likely. For example, we know that a soccer score of 1–1 is quite common, while a score of 11–15 is very, very rare. Because all outcomes are not equally likely, we cannot use the ratio between $n(E)$ and $n(S)$ to compute the theoretical probability of a team winning.

Exercise 14 – 1:

1. A learner wants to understand the term "event". So the learner rolls 2 dice hoping to get a total of 8. Which of the following is the most appropriate example of the term "event"?
   - event set $= \{(4;4)\}$
   - event set $= \{(2;6);(3;5);(4;4);(5;3);(6;2)\}$
   - event set $= \{(2;6);(6;2)\}$
2. A learner wants to understand the term "sample space". So the learner rolls a die. Which of the following is the most appropriate example of the term "sample space"?
   - $\{1;2;3;4;5;6\}$
   - $\{H;T\}$
   - $\{1;3;5\}$
3. A learner finds a 6 sided die and then rolls the die once on a table. What is the probability that the die lands on either 1 or 2?
   Write your answer as a simplified fraction.
4. A learner finds a textbook that has 100 pages. He then selects one page from the textbook. What is the probability that the page has an odd page number?
   Write your answer as a decimal (correct to 2 decimal places).
5. Even numbers from 2 to 100 are written on cards. What is the probability of selecting a multiple of 5, if a card is drawn at random?
6. A bag contains 6 red balls, 3 blue balls, 2 green balls and 1 white ball. A ball is picked at random. Determine the probability that it is:
   a) red
   b) blue or white
   c) not green
   d) not green or red
7. A playing card is selected randomly from a pack of 52 cards. Determine the probability that it is:
   a) the 2 of hearts
   b) a red card
   c) a picture card
   d) an ace
   e) a number less than 4

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.
1. 2GVY 2. 2GVZ 3. 2GW2 4. 2GW3 5. 2GW4 6. 2GW5 7. 2GW6

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14.2 Relative frequency

EMA7X

DEFINITION: *Relative frequency*

The relative frequency of an event is defined as the number of times that the event occurs during experimental trials, divided by the total number of trials conducted.

The relative frequency is not a theoretical quantity, but an experimental one. We have to repeat an experiment a number of times and count how many times the outcome of the trial is in the event set. Because it is experimental, it is possible to get a different relative frequency every time that we repeat an experiment.

VISIT:
The following video explains the concept of relative frequency using the throw of a dice.
See video: 2GW7 at www.everythingmaths.co.za

Worked example 2: Relative frequency and theoretical probability

QUESTION

We toss a coin 30 times and observe the outcomes. The results of the trials are shown in the table below.

trial	1	2	3	4	5	6	7	8	9	10
outcome	H	T	T	T	H	T	H	H	H	T
trial	11	12	13	14	15	16	17	18	19	20
outcome	H	T	T	H	T	T	T	H	T	T
trial	21	22	23	24	25	26	27	28	29	30
outcome	H	H	H	T	H	T	H	T	T	T

What is the relative frequency of observing heads after each trial and how does it compare to the theoretical probability of observing heads?

SOLUTION

Step 1: Count the number of positive outcomes

A positive outcome is when the outcome is in our event set. The table below shows a running count (after each trial t) of the number of positive outcomes p we have observed. For example, after $t = 20$ trials we have observed heads 8 times and tails 12 times and so the positive outcome count is $p = 8$.

t	1	2	3	4	5	6	7	8	9	10
p	1	1	1	1	2	2	3	4	5	5
t	11	12	13	14	15	16	17	18	19	20
p	6	6	6	7	7	7	7	8	8	8
t	21	22	23	24	25	26	27	28	29	30
p	9	10	11	11	12	12	13	13	13	13

Step 2: Compute the relative frequency

Since the relative frequency is defined as the ratio between the number of positive trials and the total number of trials,

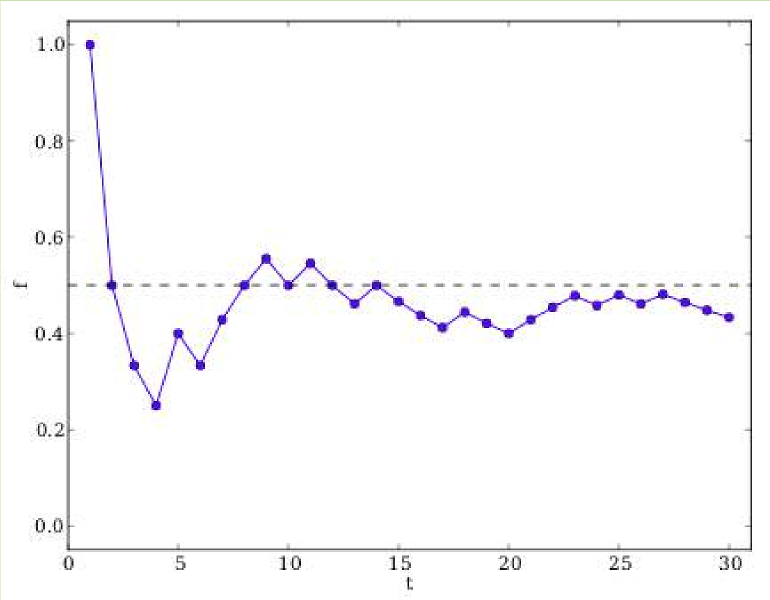
$$f = \frac{p}{t}$$

The relative frequency of observing heads, f, after having completed t coin tosses is:

t	1	2	3	4	5	6	7	8	9	10
f	1,00	0,50	0,33	0,25	0,40	0,33	0,43	0,50	0,56	0,50
t	11	12	13	14	15	16	17	18	19	20
f	0,55	0,50	0,46	0,50	0,47	0,44	0,41	0,44	0,42	0,40
t	21	22	23	24	25	26	27	28	29	30
f	0,43	0,45	0,48	0,46	0,48	0,46	0,48	0,46	0,45	0,43

From the last entry in this table we can now easily read the relative frequency after 30 trials, namely $\frac{13}{30} = 0{,}4\dot{3}$. The relative frequency is close to the theoretical probability of 0,5. In general, the relative frequency of an event tends to get closer to the theoretical probability of the event as we perform more trials.

A much better way to summarise the table of relative frequencies is in a graph:



The graph above is the plot of the relative frequency of observing heads, f, after having completed t coin tosses. It was generated from the table of numbers above by plotting the number of trials that have been completed, t, on the x-axis and the relative frequency, f, on the y-axis. In the beginning (after a small number of trials) the relative frequency fluctuates a lot around the theoretical probability at 0,5, which is shown with a dashed line. As the number of trials increases, the relative frequency fluctuates less and gets closer to the theoretical probability.

Worked example 3: Relative frequency and theoretical probability

QUESTION

While watching 10 soccer games where Team 1 plays against Team 2, we record the following final scores:

Trial	1	2	3	4	5	6	7	8	9	10
Team 1	2	0	1	1	1	1	1	0	5	3
Team 2	0	2	2	2	2	1	1	0	0	0

What is the relative frequency of Team 1 winning?

SOLUTION

Step 1:

In this experiment, each trial takes the form of Team 1 playing a soccer match against Team 2.

Step 2: Count the number of positive outcomes

We are interested in the event where Team 1 wins. From the table above we see that this happens 3 times.

Step 3: Compute the relative frequency

The total number of trials is 10. This means that the relative frequency of the event is

$$\frac{3}{10} = 0{,}3$$

It is important to understand the difference between the theoretical probability of an event and the observed relative frequency of the event in experimental trials. The theoretical probability is a number that we can compute if we have enough information about the experiment. If each possible outcome in the sample space is equally likely, we can count the number of outcomes in the event set and the number of outcomes in the sample space to compute the theoretical probability.

The relative frequency depends on the sequence of outcomes that we observe while doing a statistical experiment. The relative frequency can be different every time we redo the experiment. The more trials we run during an experiment, the closer the observed relative frequency of an event will get to the theoretical probability of the event.

So why do we need statistical experiments if we have theoretical probabilities? In some cases, like our soccer experiment, it is difficult or impossible to compute the theoretical probability of an event. Since we do not know exactly how likely it is that one soccer team will score goals against another, we can never compute the theoretical probability of events in soccer. In such cases we can still use the relative frequency to estimate the theoretical probability, by running experiments and counting the number of positive outcomes.

VISIT:

You can use this Phet simulation on probability to do some experiments with dropping a ball through a triangular grid.

Exercise 14 – 2:

1. A die is tossed 44 times and lands 5 times on the number 3.
   What is the relative frequency of observing the die land on the number 3? Write your answer correct to 2 decimal places.
2. A coin is tossed 30 times and lands 17 times on heads.
   What is the relative frequency of observing the coin land on heads? Write your answer correct to 2 decimal places.
3. A die is tossed 27 times and lands 6 times on the number 6.
   What is the relative frequency of observing the die land on the number 6? Write your answer correct to 2 decimal places.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GW8 2. 2GW9 3. 2GWB

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14.3 Venn diagrams

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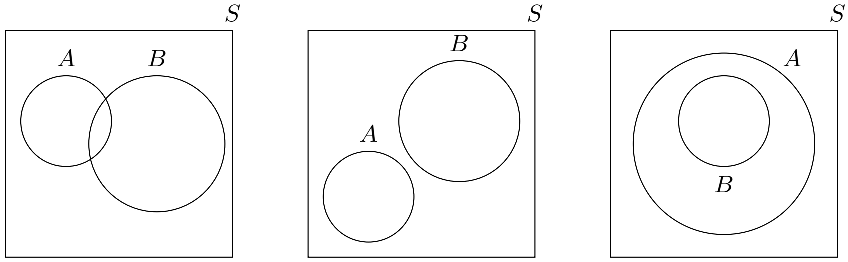
A Venn diagram is a graphical way of representing the relationships between sets. In each Venn diagram a set is represented by a closed curve. The region inside the curve represents the elements that belong to the set, while the region outside the curve represents the elements that are excluded from the set.

Venn diagrams are helpful for thinking about probability since we deal with different sets. Consider two events, A and B, in a sample space S. The diagram below shows the possible ways in which the event sets can overlap, represented using Venn diagrams:



The sets are represented using a rectangle for S and circles for each of A and B. In the first diagram the two events overlap partially. In the second diagram the two events do not overlap at all. In the third diagram one event is fully contained in the other. Note that events will always appear inside the sample space since the sample space contains all possible outcomes of the experiment.

VISIT:
This video shows how to draw a Venn diagram using a deck of cards as the sample space.
See video: 2GWC at www.everythingmaths.co.za

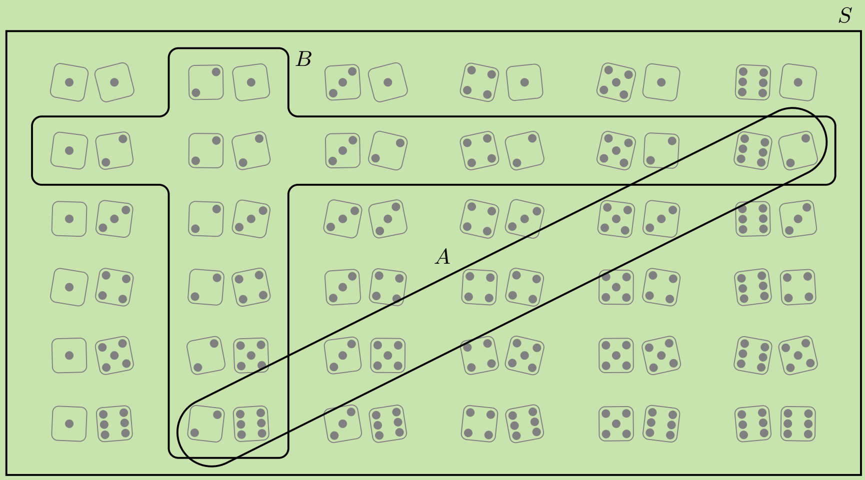
Worked example 4: Venn diagrams

QUESTION

Represent the sample space of two rolled dice and the following two events using a Venn diagram:

- Event A: the sum of the dice equals 8
- Event B: at least one of the dice shows a 2

SOLUTION



Worked example 5: Venn diagrams

QUESTION

Consider the set of diamonds removed from a deck of cards. A random card is selected from the set of diamonds.

- Write down the sample space, S, for the experiment.
- What is the value of $n(S)$?
- Consider the following two events:
  - P: An even diamond is chosen
  - R: A royal diamond is chosen

  Represent the sample space S and events P and R using a Venn diagram.

SOLUTION

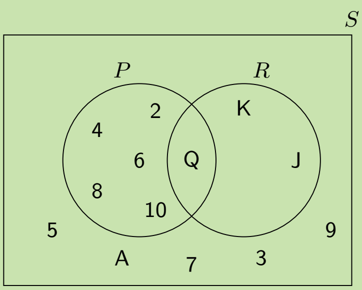
Step 1: Write down the sample space S

$S = \{A; 2; 3; 4; 5; 6; 7; 8; 9; 10; J; Q; K\}$

Step 2: Write down the value of $n(S)$

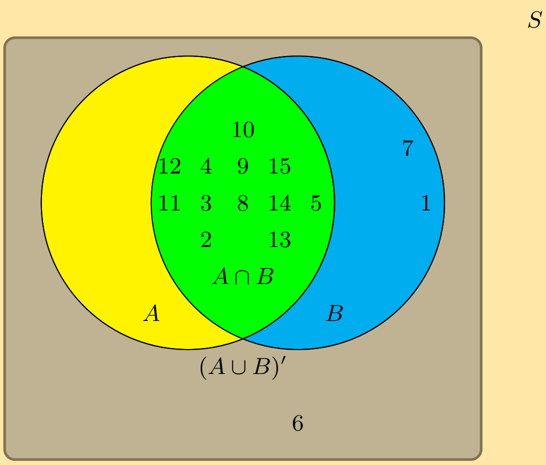
$n(S) = 13$

Step 3: Draw the Venn diagram



Exercise 14 – 3:

1. A group of learners are given the following Venn diagram:

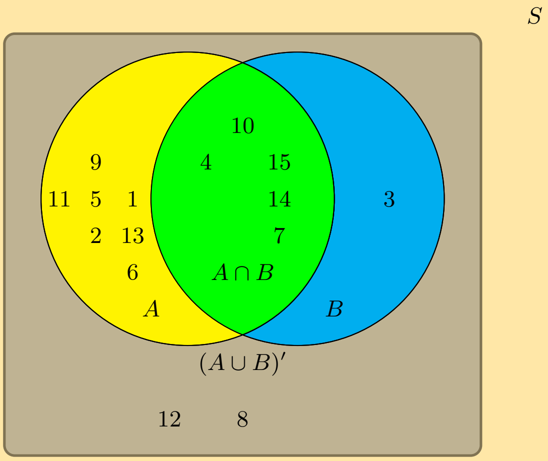


The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.

They are asked to identify the event set of B. They get stuck, and you offer to help them find it. Which of the following sets best describes the event set of B?

- $\{2; 3; 4; 5; 8; 9; 10; 11; 12; 13; 14; 15\}$
- $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 14; 15\}$
- $\{1; 6; 7\}$
- $\{6\}$

2. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the event set of A. They get stuck, and you offer to help them find it. Which of the following sets best describes the event set of A?

- $\{1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; 14; 15\}$
- $\{3; 8; 12\}$
- $\{3; 4; 7; 10; 14; 15\}$
- $\{1; 2; 4; 5; 6; 7; 9; 10; 11; 13; 14; 15\}$
- $\{4; 7; 10; 14; 15\}$

3. Pieces of paper labelled with the numbers 1 to 12 are placed in a box and the box is shaken. One piece of paper is taken out and then replaced.
   a) What is the sample space, S?
   b) Write down the set A, representing the event of taking a piece of paper labelled with a factor of 12.
   c) Write down the set B, representing the event of taking a piece of paper labelled with a prime number.
   d) Represent A, B and S by means of a Venn diagram.
   e) Find:
      i. $n(S)$
      ii. $n(A)$
      iii. $n(B)$
4. Let S denote the set of whole numbers from 1 to 16, X denote the set of even numbers from 1 to 16 and Y denote the set of prime numbers from 1 to 16. Draw a Venn diagram depicting S, X and Y.
5. There are 71 Grade 10 learners at school. All of these take some combination of Maths, Geography and History. The number who take Geography is 41, those who take History is 36, and 30 take Maths. The number who take Maths and History is 16; the number who take Geography and History is 6, and there are 8 who take Maths only and 16 who take History only.

a) Draw a Venn diagram to illustrate all this information.
b) How many learners take Maths and Geography but not History?
c) How many learners take Geography only?
d) How many learners take all three subjects?

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.
1. 2GWD 2. 2GWF 3. 2GWG 4. 2GWH 5. 2GWJ

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14.4 Union and intersection EMA7Z

DEFINITION: *Union*

The union of two sets is a new set that contains all of the elements that are in at least one of the two sets. The union is written as $A \cup B$ or "A or B".

DEFINITION: *Intersection*

The intersection of two sets is a new set that contains all of the elements that are in both sets. The intersection is written as $A \cap B$ or "A and B".

The figure below shows the union and intersection for different configurations of two events in a sample space, using Venn diagrams.

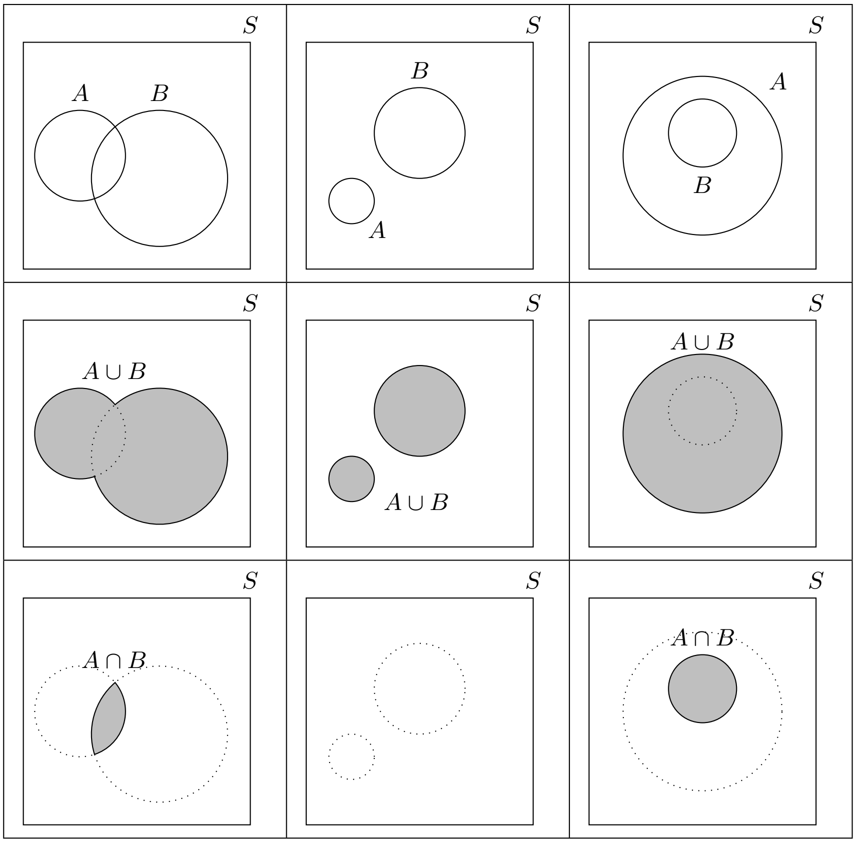
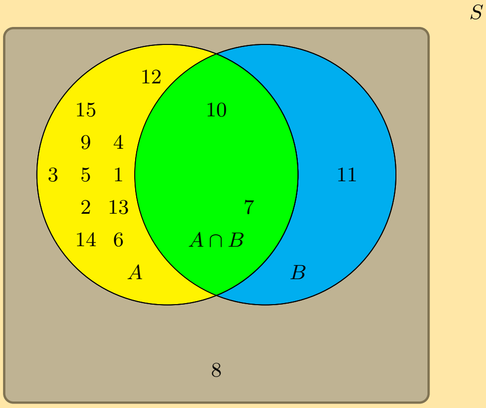


Figure 14.2: The unions and intersections of different events. Note that in the middle column the intersection, $A \cap B$, is empty since the two sets do not overlap. In the final column the union, $A \cup B$, is equal to A and the intersection, $A \cap B$, is equal to B since B is fully contained in A.

Exercise 14 – 4:

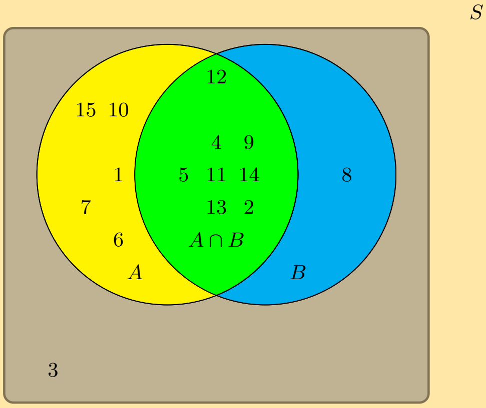
1. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the event set of the intersection between event set A and event set B, also written as $A \cap B$. They get stuck, and you offer to help them find it.
Which set best describes the event set of $A \cap B$?

   - $\{7; 10; 11\}$
   - $\{1; 2; 3; 4; 5; 6; 7; 9; 10; 11\}$
   - $\{1; 2; 3; 4; 5; 6; 7; 9; 10\}$
   - $\{7; 10\}$

2. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$
They are asked to identify the event set of the union between event set A and event set B, also written as $A \cup B$. They get stuck, and you offer to help them find it.
Which set best describes the event set of $A \cup B$?

   - $\{1; 6; 7; 10; 15\}$
   - $\{1; 2; 4; 5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15\}$
   - $\{2; 4; 5; 9; 10; 11; 12; 13; 14\}$
   - $\{3\}$

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14.5 Probability identities EMA82

By definition, the sample space contains all possible outcomes of an experiment. So we know that the probability of observing an outcome from the sample space is 1.

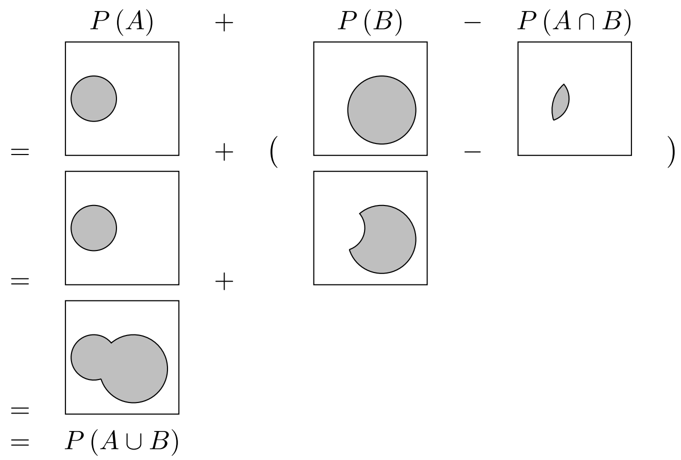
$$P(S) = 1$$

We can calculate the probability of the union of two events using:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We will prove this identity using the Venn diagrams given above.
For each of the 4 terms in the union and intersection identity, we can draw the Venn diagram and then add and subtract the different diagrams. The area of a region represents its probability.

We will do this for the first column of the Venn diagram figure given previously. You should also try it for the other columns.



VISIT:
This video gives an example of how we can add probabilities together.
See video: 2GWN at www.everythingmaths.co.za

Worked example 6: Union and intersection of events

QUESTION

Relate the probabilities of events A and B from Example 4 (two rolled dice) and show that they satisfy the identity:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

SOLUTION

Step 1: Write down the probabilities of the two events, their union and their intersection
From the Venn diagram in Example 4, we can count the number of outcomes in each event. To get the probability of an event, we divide the size of the event by the size of the sample space, which is $n(S) = 36$.

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} &&= \frac{5}{36} \\ P(B) &= \frac{n(B)}{n(S)} &&= \frac{11}{36} \\ P(A\cap B) &= \frac{n(A\cap B)}{n(S)} &&= \frac{2}{36} \\ P(A\cup B) &= \frac{n(A\cup B)}{n(S)} &&= \frac{14}{36} \end{aligned}$$

Step 2: Write down and check the identity

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$\begin{aligned} \text{RHS} &= \frac{14}{36} \\ \text{LHS} &= \frac{5}{36} + \frac{11}{36} - \frac{2}{36} \\ &= \frac{5}{36} + \frac{9}{36} \\ &= \frac{14}{36} \\ \therefore \text{RHS} &= \text{LHS} \end{aligned}$$

Exercise 14 – 5:

1. A group of learners is given the following event sets:

Event Set A	1 2 5 6		Event Set B	3		Event Set $A \cap B$	empty

   The sample space can be described as $\{n : n \,\epsilon\, \mathbb{Z},\ 1 \leq n \leq 6\}$.
   They are asked to calculate the value of $P(A\cup B)$. They get stuck, and you offer to calculate it for them. Give your answer as a decimal number, rounded to two decimal places.

2. A group of learners is given the following event sets:

Event Set A	1 2 6		Event Set B	1 5		Event Set $A \cup B$	1 2 5 6

   The sample space can be described as $\{n : n \,\epsilon\, \mathbb{Z},\ 1 \leq n \leq 6\}$.
   They are asked to calculate the value of $P(A\cap B)$. They get stuck, and you offer to calculate it for them. Give your answer as a decimal number, rounded to two decimal value.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'. 1. 2GWP 2. 2GWQ

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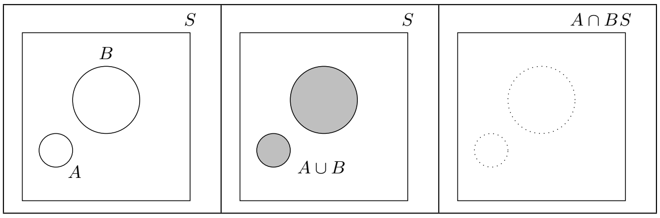
DEFINITION: *Mutually exclusive events*

Two events are called mutually exclusive if they cannot occur at the same time. Whenever an outcome of an experiment is in the first event it cannot also be in the second event, and vice versa.

Another way of saying this is that the two event sets, A and B, cannot have any elements in common, or $P(A \cap B) = \varnothing$ (where $\varnothing$ denotes the empty set). We have already seen the Venn diagrams of mutually exclusive events in the middle column of the Venn diagrams provided earlier.



From this figure you can see that the intersection has no elements. You can also see that the probability of the union is the sum of the probabilities of the events.

$$P(A \cup B) = P(A) + P(B)$$

This relationship is true for mutually exclusive events only.

Worked example 7: Mutually exclusive events

QUESTION

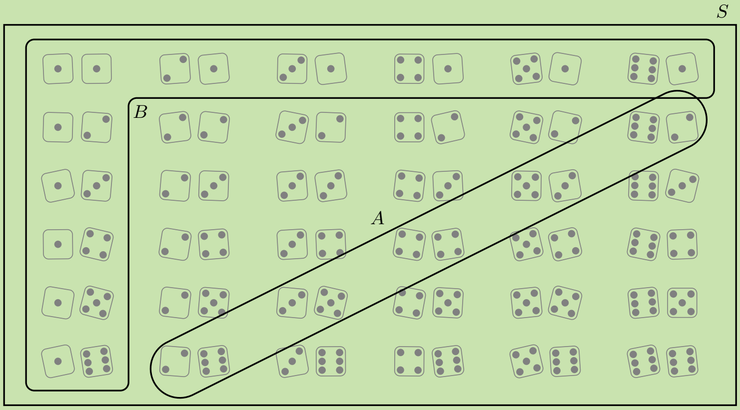
We roll two dice and are interested in the following two events:

- A : The sum of the dice equals 8
- B : At least one of the dice shows a 1

Show that the events are mutually exclusive.

SOLUTION

Step 1: Draw the sample space and the two events



Step 2: Determine the intersection

From the above figure we notice that there are no elements in common in A and B. Therefore the events are mutually exclusive.

Exercise 14 – 6:

State whether the following events are mutually exclusive or not.

1. A fridge contains orange juice, apple juice and grape juice. A cooldrink is chosen at random from the fridge. Event A: the cooldrink is orange juice. Event B: the cooldrink is apple juice.
2. A packet of cupcakes contains chocolate cupcakes, vanilla cupcakes and red velvet cupcakes. A cupcake is chosen at random from the packet. Event A: the cupcake is red velvet. Event B: the cupcake is vanilla.
3. A card is chosen at random from a deck of cards. Event A: the card is a red card. Event B: the card is a picture card.
4. A cricket team plays a game. Event A: they win the game. Event B: they lose the game.

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1. 2GWR 2. 2GWS 3. 2GWT 4. 2GWV

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14.7 Complementary events EMA84

DEFINITION: *Complementary set*

The complement of a set, A, is a new set that contains all of the elements that are not in A. We write the complement of A as A', or sometimes not (A).

For an experiment with sample space S and an event A we can derive some identities for complementary events. Since every element in A is not in A', we know that complementary events are mutually exclusive.

$$A \cap A' = \varnothing$$

Since every element in the sample space is either in A or in A', the union of complementary events covers the sample space.

$$A \cup A' = S$$

From the previous two identities, we also know that the probabilities of complementary events sum to 1.

$$P(A) + P(A') = P(A \cup A') = P(S) = 1$$

Worked example 8: Reasoning with Venn diagrams

QUESTION

In a survey 70 people were questioned about which product they use: A or B or both. The report of the survey shows that 25 people use product A, 35 people use product B and 15 people use neither. Use a Venn diagram to work out how many people:

1. use product A only
2. use product B only
3. use both product A and product B

SOLUTION

Step 1: Summarise the sizes of the sample space, the event sets, their union and their intersection

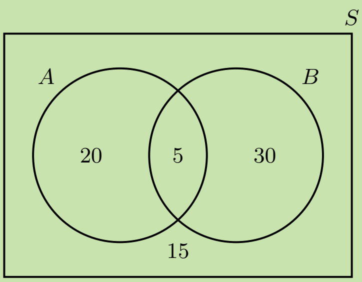
- We are told that 70 people were questioned, so the size of the sample space is $n(S) = 70$.
- We are told that 25 people use product A, so $n(A) = 25$.
- We are told that 35 people use product B, so $n(B) = 35$.
- We are told that 15 people use neither product. This means that $70 - 15 = 55$ people use at least one of the two products, so $n(A \cup B) = 55$.
- We are not told how many people use both products, so we have to work out the size of the intersection, $A \cap B$, by using the identity for the union of two events:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \frac{n(A \cup B)}{n(S)} &= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} \\ \frac{55}{70} &= \frac{25}{70} + \frac{35}{70} - \frac{n(A \cap B)}{70} \\ \therefore n(A \cap B) &= 25 + 35 - 55 \\ &= 5 \end{aligned}$$

Step 2: Determine whether the events are mutually exclusive

Since the intersection of the events, $A \cap B$, is not empty, the events are not mutually exclusive. This means that their circles should overlap in the Venn diagram.

Step 3: Draw the Venn diagram and fill in the numbers

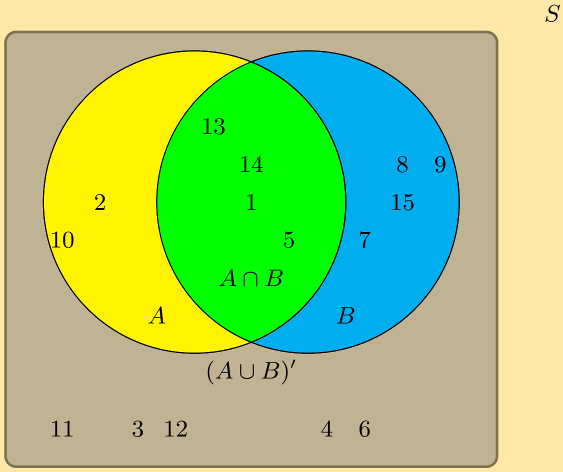


Step 4: Read off the answers

1. 20 people use product A only.
2. 30 people use product B only.
3. 5 people use both products.

Exercise 14 – 7:

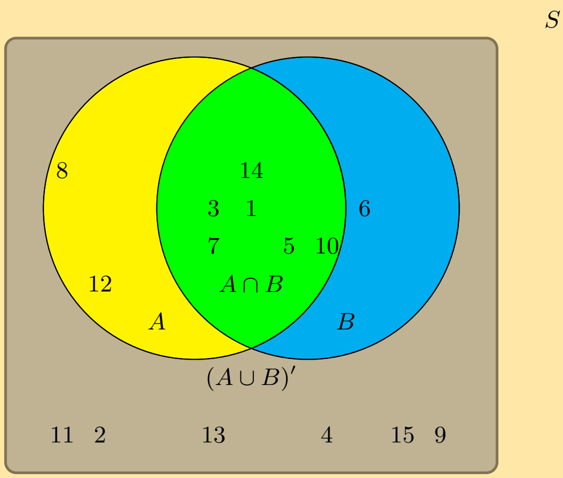
1. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the complementary event set of B, also known as B'. They get stuck, and you offer to help them find it.
Which of the following sets best describes the event set of B'?

- $\{1; 5; 13; 14\}$
- $\{2; 3; 4; 6; 10; 11; 12\}$
- $\{3; 4; 6; 11; 12\}$

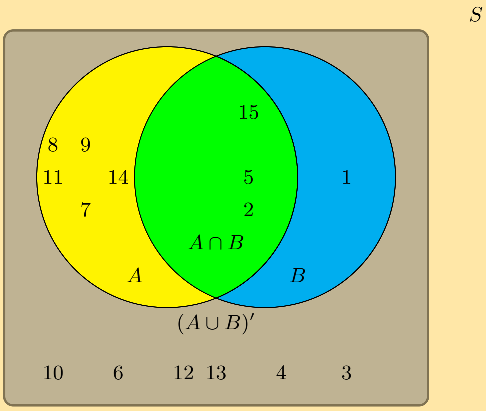
2. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the complementary event set of $(A \cup B)$, also known as $(A \cup B)'$. They get stuck, and you offer to help them find it.
Which of the following sets best describes the event set of $(A \cup B)'$?

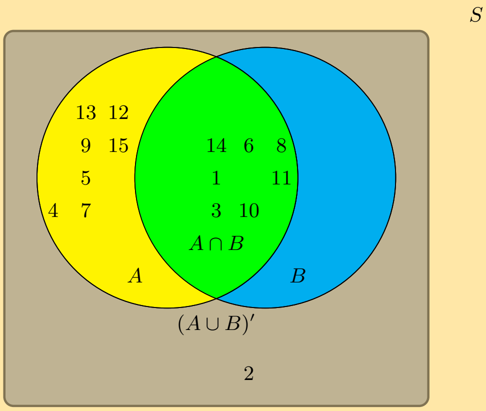
- $\{2; 4; 9; 11; 13; 15\}$
- $\{1; 3; 5; 6; 7; 8; 10; 12; 14\}$
- $\{6; 8; 12\}$

3. Given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$. Are $(A \cup B)'$ and $A \cup B$ mutually exclusive?

4. Given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$. Are A' and B' mutually exclusive?

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1. 2GWW 2. 2GWX 3. 2GWY 4. 2GWZ

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14.8 Chapter summary EMA85

See presentation: 2GX2 at www.everythingmaths.co.za

- An experiment refers to an uncertain process.
- An outcome of an experiment is a single result of that experiment.
- The sample space of an experiment is the set of all possible outcomes of that experiment. The sample space is denoted with the symbol S and the size of the sample space (the total number of possible outcomes) is denoted with $n(S)$.
- An event is a specific set of outcomes of an experiment that you are interested in. An event is denoted with the letter E and the number of outcomes in the event with $n(E)$.
- A probability is a real number between 0 and 1 that describes how likely it is that an event will occur.

- A probability of 0 means that an event will never occur.
- A probability of 1 means that an event will always occur.
- A probability of 0,5 means that an event will occur half the time, or 1 time out of every 2.

- A probability can also be written as a percentage or as a fraction.
- When all of the possible outcomes of an experiment have an equal chance of occurring, we can compute the exact theoretical probability of an event. The probability of an event is the ratio between the number of outcomes in the event set and the number of possible outcomes in the sample space.

$$P(E) = \frac{n(E)}{n(S)}$$

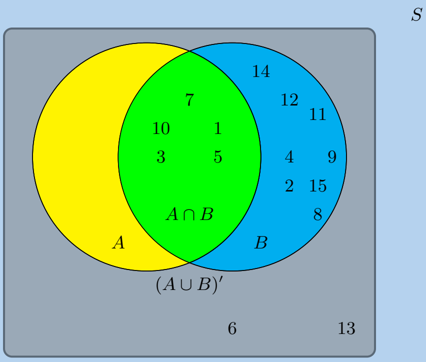
- The relative frequency of an event is defined as the number of times that the event occurs during experimental trials, divided by the total number of trials conducted.

$$f = \frac{\text{number of positive trials}}{\text{number of trials}} = \frac{p}{n}$$

- The union of two sets is a new set that contains all of the elements that are in at least one of the two sets. The union is written as $A \cup B$ or A or B.
- The intersection of two sets is a new set that contains all of the elements that are in both sets. The intersection is written as $A \cap B$ or A and B.
- The probability of observing an outcome from the sample space is 1: $P(S) = 1$.
- The probability of the union of two events is calculated using: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Mutually exclusive events are two events that cannot occur at the same time. Whenever an outcome of an experiment is in the first event, it can not also be in the second event.
- The complement of a set, A, is a different set that contains all of the elements that are not in A. We write the complement of A as A' or "not (A)".
- Complementary events are mutually exclusive: $A \cap A' = \varnothing$.
- Complementary events cover the sample space: $A \cup A' = S$
- Probabilities of complementary events sum to 1: $P(A) + P(A') = P(A \cup A') = P(S) = 1$.

End of chapter Exercise 14 – 8:

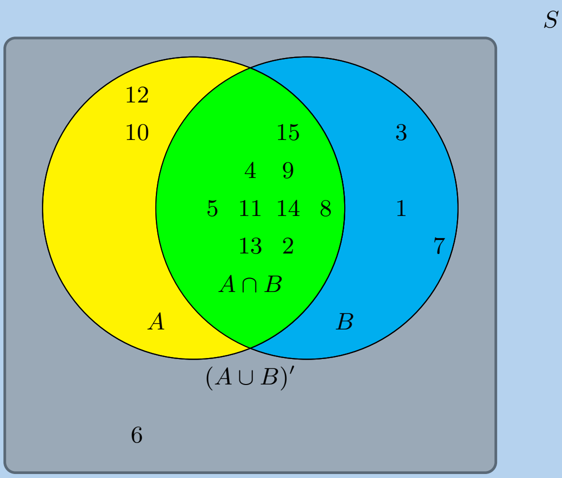
1. A learner wants to understand the term "outcome". So the learner rolls a die. Which of the following is the most appropriate example of the term "outcome"?
   - A teacher walks into the class room.
   - The die lands on the number 5.
   - The clock strikes 3 pm.
2. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the event set of B. They get stuck, and you offer to help them find it.
Which of the following sets best describes the event set of B?

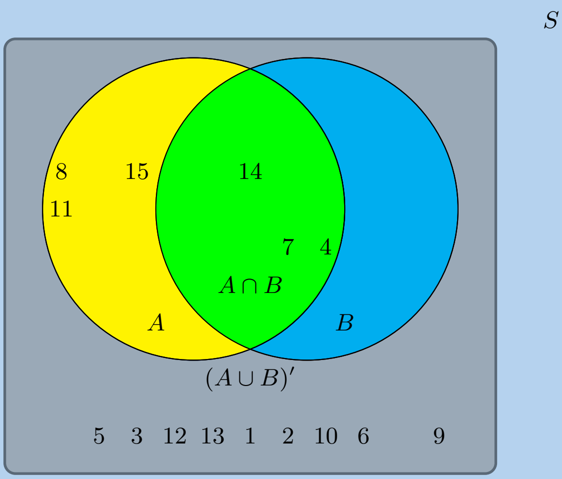
- $\{6; 13\}$
- $\{1; 3; 5; 7; 10\}$
- $\{2; 4; 6; 8; 9; 11; 12; 13; 14; 15\}$
- $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 14; 15\}$

3. A group of learners are given the following Venn diagram:



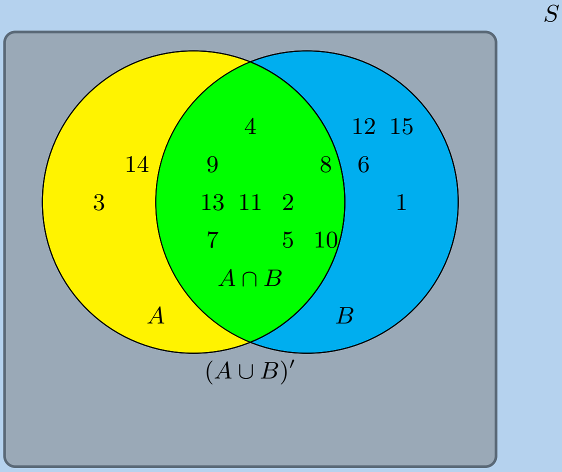
The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.
They are asked to identify the event set of the union between event set A and event set B, also written as $A \cup B$. They get stuck, and you offer to help them find it.
Write down the event set that best describes $A \cup B$.

4. Given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.
Are $A \cup B$ and $(A \cup B)'$ mutually exclusive?

5. A group of learners are given the following Venn diagram:



The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 15\}$.

They are asked to identify the complementary event set of $(A \cap B)$, also known as $(A \cap B)'$. They get stuck, and you offer to help them find it.

Write down the set that best describes the event set of $(A \cap B)'$.

6. A learner finds a deck of 52 cards and then takes one card from the deck. What is the probability that the card is a king?

   Write your answer as a decimal (correct to 2 decimal places).

7. A die is tossed 21 times and lands 2 times on the number 3.

   What is the relative frequency of observing the die land on the number 3? Write your answer correct to 2 decimal places.

8. A coin is tossed 44 times and lands 22 times on heads.

   What is the relative frequency of observing the coin land on heads? Write your answer correct to 2 decimal places.

9. A group of 45 children were asked if they eat Frosties, Strawberry Pops or both. 31 children said they eat both and 6 said they only eat Frosties. What is the probability that a child chosen at random will eat only Strawberry Pops?

10. In a group of 42 learners, all but 3 had a packet of chips or a cooldrink or both. If 23 had a packet of chips and 7 of these also had a cooldrink, what is the probability that one learner chosen at random has:

    a) both chips and cooldrink
    b) only cooldrink

11. A box contains coloured blocks. The number of each colour is given in the following table.

Colour	Purple	Orange	White	Pink
Number of blocks	24	32	41	19

    A block is selected randomly. What is the probability that the block will be:

    a) purple
    b) purple or white
    c) pink and orange
    d) not orange

12. A small nursery school has a class with children of various ages. The table gives the number of children of each age in the class.

Age	3 years old	4 years old	5 years old
Male	2	7	6
Female	6	5	4

If a child is selected at random what is the probability that the child will be:

a) a female
b) a 4 year old male
c) aged 3 or 4
d) aged 3 and 4
e) not 5
f) either 3 or female

13. Fiona has 85 labelled discs, which are numbered from 1 to 85. If a disc is selected at random what is the probability that the disc number:

    a) ends with 5
    b) is a multiple of 3
    c) is a multiple of 6
    d) is number 65
    e) is not a multiple of 5
    f) is a multiple of 3 or 4
    g) is a multiple of 2 and 6
    h) is number 1

14. Use a Venn diagram to work out the following probabilities for a die being rolled:

    a) a multiple of 5 and an odd number
    b) a number that is neither a multiple of 5 nor an odd number
    c) a number which is not a multiple of 5, but is odd

15. A packet has yellow sweets and pink sweets. The probability of taking out a pink sweet is $\frac{7}{12}$. What is the probability of taking out a yellow sweet?

16. In a car park with 300 cars, there are 190 Opels. What is the probability that the first car to leave the car park is:

    a) an Opel
    b) not an Opel

17. Nezi has 18 loose socks in a drawer. Eight of these are plain orange and two are plain pink. The remaining socks are neither orange nor pink. Calculate the probability that the first sock taken out at random is:

    a) orange
    b) not orange
    c) pink
    d) not pink
    e) orange or pink
    f) neither orange nor pink

18. A plate contains 9 shortbread cookies, 4 ginger biscuits, 11 chocolate chip cookies and 18 Jambos. If a biscuit is selected at random, what is the probability that:

    a) it is either a ginger biscuit or a Jambo
    b) it is not a shortbread cookie

19. 280 tickets were sold at a raffle. Jabulile bought 15 tickets. What is the probability that Jabulile:

   a) wins the prize

   b) does not win the prize

20. A group of children were surveyed to see how many had red hair and brown eyes. 44 children had red hair but not brown eyes, 14 children had brown eyes and red hair, 5 children had brown eyes but not red hair and 40 children did not have brown eyes or red hair.

   a) How many children were in the school?

   b) What is the probability that a child chosen at random has:

      i. brown eyes

      ii. red hair

   c) A child with brown eyes is chosen randomly. What is the probability that this child will have red hair?

21. A jar has purple sweets, blue sweets and green sweets in it. The probability that a sweet chosen at random will be purple is $\frac{1}{7}$ and the probability that it will be green is $\frac{3}{5}$.

   a) If I choose a sweet at random what is the probability that it will be:

      i. purple or blue

      ii. green

      iii. purple

   b) If there are 70 sweets in the jar how many purple ones are there?

   c) $\frac{2}{5}$ of the purple sweets in (b) have streaks on them and the rest do not. How many purple sweets have streaks?

22. Box A contains 3 cards numbered 1, 2 and 3.
Box B contains 2 cards numbered 1 and 2.
One card is removed at random from each box.
Find the probability that:

   a) the sum of the numbers is 4.

   b) the sum of the two numbers is a prime number.

   c) the product of the two numbers is at least 3.

   d) the sum is equal to the product.

23. A card is drawn at random from an ordinary pack of 52 playing cards.

   a) Find the probability that the card drawn is:

      i. the three of diamonds

      ii. the three of diamonds or any heart

      iii. a diamond or a three

   b) The card drawn is the three of diamonds. It is placed on the table and a second card is drawn. What is the probability that the second card drawn is not a diamond.

24. A group of learners is given the following event sets:

Event Set A	3	4

Event Set B	2	4	5

Event Set $A \cup B$	2	3	4	5

The sample space can be described as $\{n : n \in \mathbb{Z},\ 1 \leq n \leq 6\}$
They are asked to calculate the value of $P(A \cap B)$. They get stuck, and you offer to calculate it for them. Give your answer as a decimal number, rounded to two decimal value.

25. For each of the following, draw a Venn diagram to represent the situation and find an example to illustrate the situation.
    a) a sample space in which there are two events that are not mutually exclusive
    b) a sample space in which there are two events that are complementary
26. Use a Venn diagram to prove that the probability of either event A or B occurring (A and B are not mutually exclusive) is given by:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

27. All the clubs are taken out of a pack of cards. The remaining cards are then shuffled and one card chosen. After being chosen, the card is replaced before the next card is chosen.
    a) What is the sample space?
    b) Find a set to represent the event, P, of drawing a picture card.
    c) Find a set for the event, N, of drawing a numbered card.
    d) Represent the above events in a Venn diagram.
    e) What description of the sets P and N is suitable? (Hint: Find any elements of P in N and of N in P.)
28. A survey was conducted at Mutende Primary School to establish how many of the 650 learners buy vetkoek and how many buy sweets during break. The following was found:
    - 50 learners bought nothing
    - 400 learners bought vetkoek
    - 300 learners bought sweets

    a) Represent this information with a Venn diagram.
    b) If a learner is chosen randomly, calculate the probability that this learner buys:
        i. sweets only
        ii. vetkoek only
        iii. neither vetkoek nor sweets
        iv. vetkoek and sweets
        v. vetkoek or sweets
29. In a survey at Lwandani Secondary School, 80 people were questioned to find out how many read the Sowetan, how many read the Daily Sun and how many read both. The survey revealed that 45 read the Daily Sun, 30 read the Sowetan and 10 read neither. Use a Venn diagram to find the percentage of people that read:
    a) only the Daily Sun
    b) only the Sowetan
    c) both the Daily Sun and the Sowetan
30. In a class there are
    - 8 learners who play football and hockey
    - 7 learners who do not play football or hockey
    - 13 learners who play hockey
    - 19 learners who play football

    How many learners are there in the class?
31. Of 36 people, 17 have an interest in reading magazines and 12 have an interest in reading books, 6 have an interest in reading both magazines and books.

a) Represent the information in a Venn diagram.

b) How many people have no interest in reading magazines or books?

c) If a person is chosen at random from the group, find the probability that the person will:

i. have an interest in reading magazines and books.

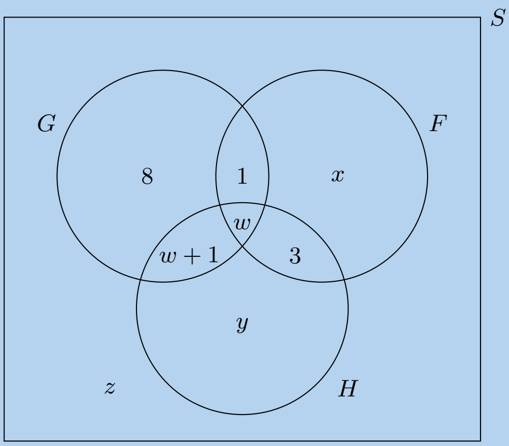
ii. have an interest in reading books only.

iii. not have any interest in reading books.

32. 30 learners were surveyed and the following information was revealed from this group:

- 18 learners take Geography
- 10 learners take French
- 6 learners take History, but take neither Geography nor French.

In addition the following Venn Diagram has been filled in below:
Let G be the event that a learner takes Geography.
Let F be the event that a learner takes French.
Let H be the event that a learner takes History.



a) From the information above, determine the values of w, x, y and z.

b) Determine the probability that a learner chosen at random from this group:

i. takes only Geography.

ii. takes French and History, but not Geography.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2GX3	2. 2GX4	3. 2GX5	4. 2GX6	5. 2GX7	6. 2GX8	7. 2GX9	8. 2GXB
9. 2GXC	10. 2GXD	11. 2GXF	12. 2GXG	13. 2GXH	14. 2GXJ	15. 2GXK	16. 2GXM
17. 2GXN	18. 2GXP	19. 2GXQ	20. 2GXR	21. 2GXS	22. 2GXT	23. 2GXV	24. 2GXW
25a. 2GXX	25b. 2GXY	26. 2GXZ	27. 2GY2	28. 2GY3	29. 2GY4	30. 2GY5	31. 2GY6
32. 2GY7							

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