CHAPTER 2

Functions

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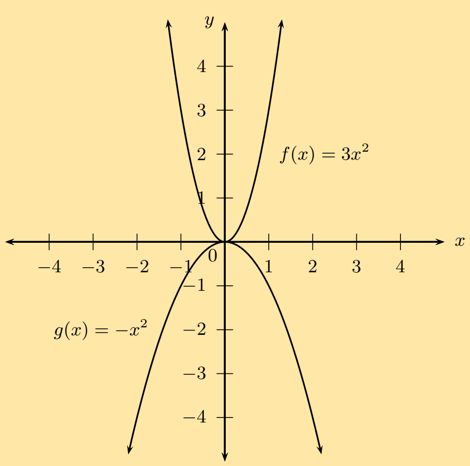
- Learners must be encouraged to check whether or not the inverse is a function.
- It is very important that learners understand that $f^{-1}(x)$ notation can only be used if the inverse is a function.
- Learners must not confuse the inverse function f^{-1} and the reciprocal $\frac{1}{f(x)}$.
- Encourage learners to state restrictions, particularly for quadratic functions.
- Learners must understand that $y = \sqrt{-x}$ has real solutions for $x < 0$.
- Exercises on parabolic functions with horizontal and vertical shifts have been included for enrichment only and are clearly marked.
- The logarithmic function is introduced as the inverse of the exponential function. Learners need to understand that the logarithmic function allows us to rewrite an exponential expression with the exponent as the subject of the formula.
- It is very important that learners can change from exponential form to logarithmic form and vice versa. This skill is also important for finding the period of an investment or loan in the Finance chapter.
- Learners should be encouraged to use the definition and change of base to solve problems. Manipulation involving the logarithmic laws is not examinable.
- Learners should be encouraged to be familiar with the LOG function on their calculator and also to use their calculator to check answers.
- Enrichment content is not examinable and is clearly marked.

2.1 Revision

Exercise 2 – 1: Revision

1. Sketch the graphs on the same set of axes and determine the following for each function:
    - Intercepts
    - Turning point
    - Axes of symmetry
    - Domain and range
    - Maximum and minimum values

    a) $f(x) = 3x^2$ and $g(x) = -x^2$
    Solution:



For $f(x)$:

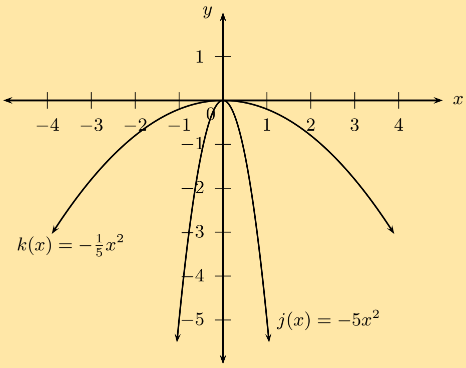
$$\begin{aligned}\text{Intercept: } & (0;0)\\ \text{Turning point: } & (0;0)\\ \text{Axes of symmetry: } & x=0\\ \text{Domain: } & \{x : x \in \mathbb{R}\}\\ \text{Range: } & \{y : y \geq 0, y \in \mathbb{R}\}\\ \text{Minimum value: } & y=0\end{aligned}$$

For $g(x)$:

$$\begin{aligned}\text{Intercept: } & (0;0)\\ \text{Turning point: } & (0;0)\\ \text{Axes of symmetry: } & x=0\\ \text{Domain: } & \{x : x \in \mathbb{R}\}\\ \text{Range: } & \{y : y \leq 0, y \in \mathbb{R}\}\\ \text{Maximum value: } & y=0\end{aligned}$$

b) $j(x) = -\frac{1}{5}x^2$ and $k(x) = -5x^2$

Solution:



For $j(x)$:

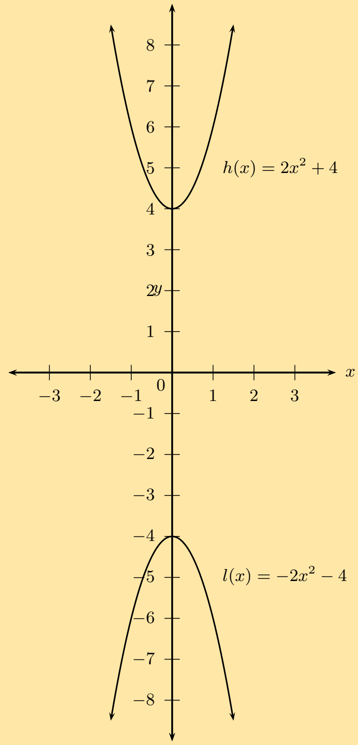
$$\begin{aligned}\text{Intercepts: } & (0;0) \;\; (0;0)\\ \text{Turning point: } & (0;0)\\ \text{Axes of symmetry: } & x=0\\ \text{Domain: } & \{x : x \in \mathbb{R}\}\\ \text{Range: } & \{y : y \leq 0, y \in \mathbb{R}\}\\ \text{Maximum value: } & y=0\end{aligned}$$

For $k(x)$:

$$\begin{aligned}\text{Intercepts: } & (0;0) \;\; (0;0)\\ \text{Turning point: } & (0;0)\\ \text{Axes of symmetry: } & x=0\\ \text{Domain: } & \{x : x \in \mathbb{R}\}\\ \text{Range: } & \{y : y \leq 0, y \in \mathbb{R}\}\\ \text{Maximum value: } & y=0\end{aligned}$$

c) $h(x) = 2x^2 + 4$ and $l(x) = -2x^2 - 4$

Solution:



For $h(x)$:

$$\begin{aligned}
\text{Intercept: } & (0;4) \\
\text{Turning point: } & (0;4) \\
\text{Axes of symmetry: } & x = 0 \\
\text{Domain: } & \{x : x \in \mathbb{R}\} \\
\text{Range: } & \{y : y \geq 4, y \in \mathbb{R}\} \\
\text{Minimum value: } & y = 4
\end{aligned}$$

For $l(x)$:

$$\begin{aligned}
\text{Intercept: } & (0;-4) \\
\text{Turning point: } & (0;-4) \\
\text{Axes of symmetry: } & x = 0 \\
\text{Domain: } & \{x : x \in \mathbb{R}\} \\
\text{Range: } & \{y : y \leq -4, y \in \mathbb{R}\} \\
\text{Maximum value: } & y = -4
\end{aligned}$$

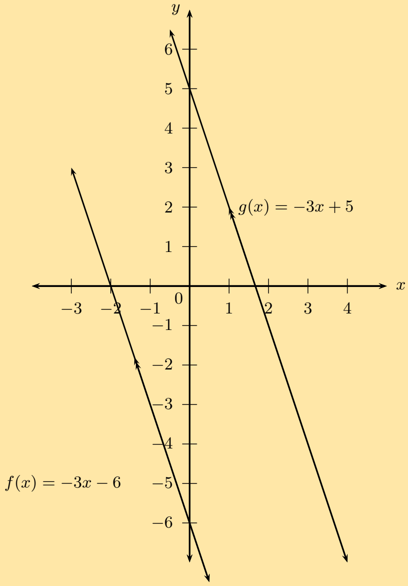
2. Given $f(x) = -3x - 6$ and $g(x) = mx + c$. Determine the values of m and c if $g \parallel f$ and g passes through the point $(1;2)$. Sketch both functions on the same system of axes.

   Solution:

$$\begin{aligned}
g(x) &= mx + c \\
m &= -3 \\
g(x) &= -3x + c \\
\text{Substitute } (1;2) \quad 2 &= -3(1) + c \\
\therefore c &= 5 \\
\therefore g(x) &= -3x + 5
\end{aligned}$$

$$\text{Intercepts for } g: \quad (\frac{5}{3};0);(0;5)$$

$$\text{Intercepts for } f: \quad (-2;0);(0;-6)$$



3. Given $m : \frac{x}{2} - \frac{y}{3} = 1$ and $n : -\frac{y}{3} = 1$. Determine the x- and y-intercepts and sketch both graphs on the same system of axes.

Solution:

For $m(x)$:

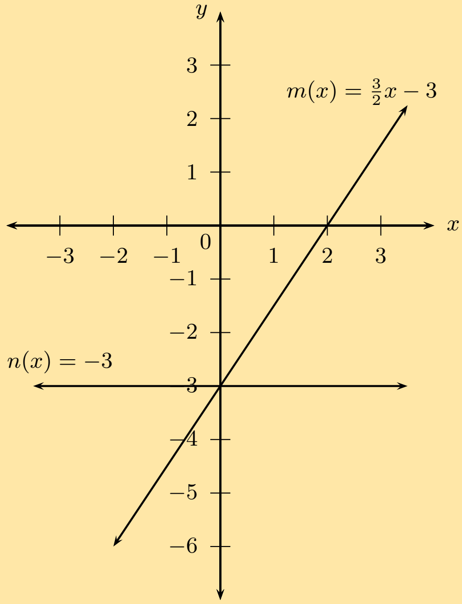
$$\begin{aligned} \frac{x}{2} - \frac{y}{3} &= 1 \\ \text{Let } x = 0: \quad -\frac{y}{3} &= 1 \\ y &= -3 \\ \text{Let } y = 0: \quad \frac{x}{2} &= 1 \\ x &= 2 \\ \text{Intercepts: } &(2; 0); (0; -3) \end{aligned}$$

For $n(x)$:

$$\begin{aligned} -\frac{y}{3} &= 1 \\ \therefore y &= -3 \\ \text{Intercepts: } &(0; -3) \end{aligned}$$

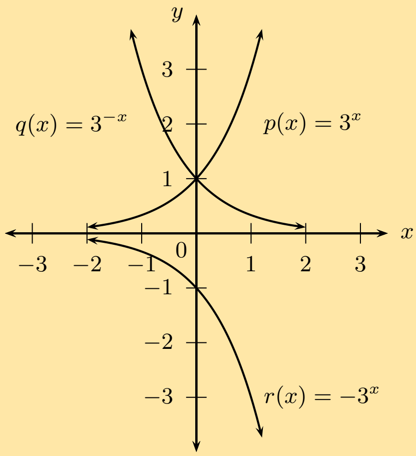
Write the linear function in standard form:

$$\begin{aligned} \frac{x}{2} - \frac{y}{3} &= 1 \\ \frac{3x}{2} - y &= 3 \\ \frac{3x}{2} - 3 &= y \\ \therefore y &= \frac{3x}{2} - 3 \end{aligned}$$



4. Given $p(x) = 3^x, q(x) = 3^{-x}$ and $r(x) = -3^x$.

   a) Sketch p, q and r on the same system of axes.
   Solution:



   b) For each of the functions, determine the intercepts, asymptotes, domain and range.
   Solution:
   For $p(x)$:

Intercept: $(0;1)$
Asymptote: $y = 0$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y > 0\}$

   For $q(x)$:

Intercept: $(0;1)$
Asymptote: $y = 0$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y > 0\}$

   For $r(x)$:

Intercept: $(0; -1)$
Asymptote: $y = 0$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y < 0\}$

Check answers online with the exercise code below or click on 'show me the answer'.
1a. 2899 1b. 289B 1c. 289C 2. 289D 3. 289F 4a. 289G
4b. 289H

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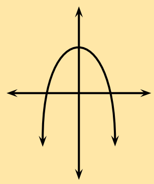
2.2 Functions and relations

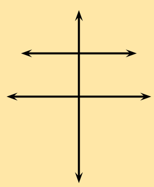
Exercise 2 – 2: Identifying functions

1. Consider the graphs given below and determine whether or not they are functions:



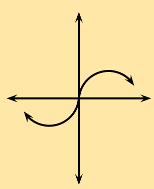
a)

Solution:
Yes



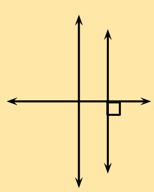
b)

Solution:
Yes



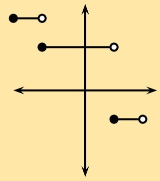
c)

Solution:
Yes

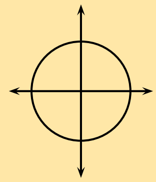


d)

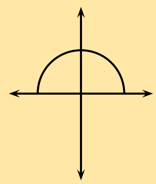
Solution:
No

e) 

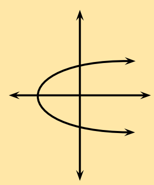
Solution:
Yes

f) 

Solution:
No

g) 

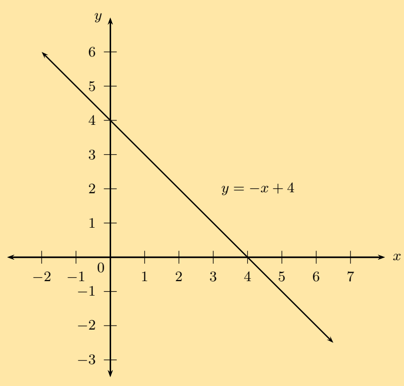
Solution:
Yes

h) 

Solution:
No

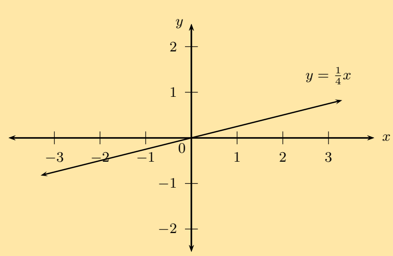
2. Sketch the following and determine whether or not they are functions:

   a) $x + y = 4$
   Solution:



   One-to-one relation: therefore is a function.

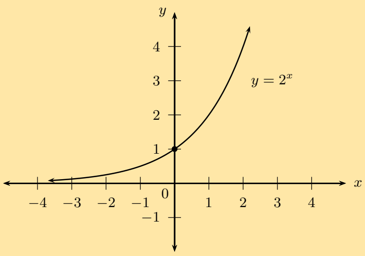
   b) $y = \frac{x}{4}$
   Solution:



One-to-one relation: therefore is a function.

c) $y = 2^x$

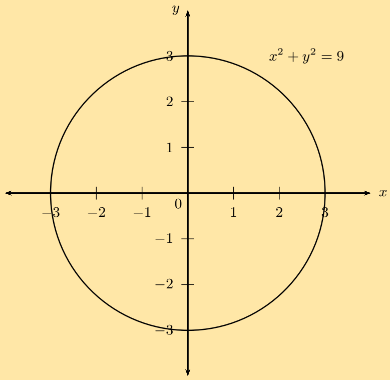
Solution:



One-to-one relation: therefore is a function.

d) $x^2 + y^2 = 9$

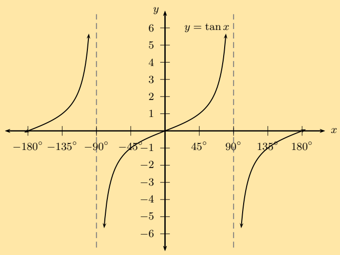
Solution:



One-to-many relation: therefore is not a function.

e) $y = \tan x$

Solution:



Many-to-one relation: therefore is a function.

3. The table below gives the average per capita income, d, in a region of the country as a function of u, the percentage of unemployed people. Write down an equation to show that the average income is a function of the percentage of unemployed people.

u	1	2	3	4
d	22 500	22 000	21 500	21 000

Solution:

Per capita income is a measure of the average amount of money earned per person in a certain area.

We see that there is a constant difference of -500 between the consecutive values of d, therefore the relation is a linear function of the form $y = mx + c$:

u is the independent variable and d is the dependent variable.

$$\begin{aligned} d &= mu + c \\ m &= -500 \\ d &= -500u + c \end{aligned}$$

Substitute any of the given set of values to solve for c:

$$\begin{aligned} 22\ 500 &= -500(1) + c \\ \therefore c &= 23\ 000 \end{aligned}$$

The function is: $d = -500u + 23\ 000$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 289J 1b. 289K 1c. 289M 1d. 289N 1e. 289P 1f. 289Q
1g. 289R 1h. 289S 2a. 289T 2b. 289V 2c. 289W 2d. 289X
2e. 289Y 3. 289Z

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2.3 Inverse functions

2.4 Linear functions

Inverse of the function $y = ax + q$

Exercise 2 – 3: Inverse of the function $y = ax + q$

1. Given $f(x) = 5x + 4$, find $f^{-1}(x)$.

   Solution:

$$\begin{aligned} f(x): y &= 5x + 4 \\ f^{-1}(x): x &= 5y + 4 \\ \therefore x - 4 &= 5y \\ y &= \frac{1}{5}x - \frac{4}{5} \\ \therefore f^{-1}(x) &= \frac{1}{5}x - \frac{4}{5} \end{aligned}$$

2. Consider the relation $f(x) = -3x - 7$.

   a) Is the relation a function? Explain your answer.

      Solution:

      It is a function. Every x-value relates to only one y-value, it is a one-to-one relation.

   b) Identify the domain and range.

      Solution:

$$\begin{aligned} \text{Domain } &\{x : x \in \mathbb{R}\} \\ \text{Range } &\{y : y \in \mathbb{R}\} \end{aligned}$$

   c) Determine $f^{-1}(x)$.

Solution:

$$
\begin{aligned}
f(x): y &= -3x - 7\\
f^{-1}(x): x &= -3y - 7\\
\therefore x + 7 &= -3y\\
y &= -\frac{1}{3}x - \frac{7}{3}\\
\therefore f^{-1}(x) &= -\frac{1}{3}x - \frac{7}{3}
\end{aligned}
$$

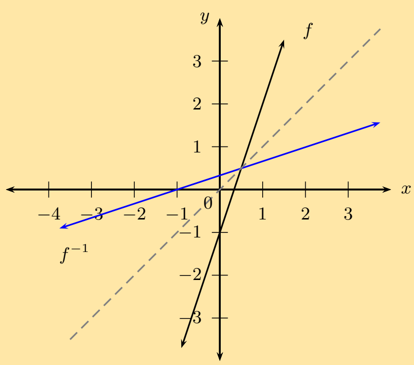
3. a) Sketch the graph of the function $f(x) = 3x - 1$ and its inverse on the same system of axes. Indicate the intercepts and the axis of symmetry of the two graphs.

Solution:

$$
\begin{aligned}
y &= 3x - 1\\
x &= 3y - 1\\
\therefore 3y &= x + 1\\
y &= \frac{1}{3}x + \frac{1}{3}
\end{aligned}
$$

The intercepts are:

$$
\begin{aligned}
f(x): x = 0, y &= -1\\
y = 0, x &= \frac{1}{3}\\
f^{-1}(x): x = 0, y &= \frac{1}{3}\\
y = 0, x &= -1
\end{aligned}
$$



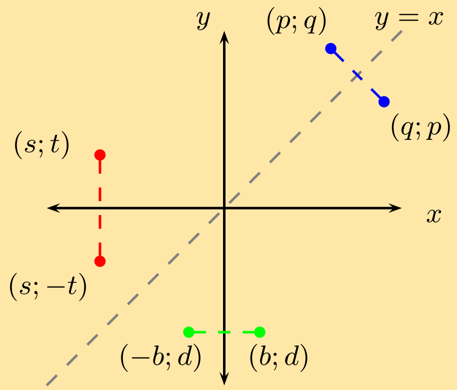
b) $T\left(\frac{4}{3}; 3\right)$ is a point on f and R is a point on f^{-1}. Determine the coordinates of R if R and T are symmetrical.

Solution: $R\left(3; \frac{4}{3}\right)$

4. a) Explain why the line $y = x$ is an axis of symmetry for a function and its inverse.

Solution:

To reflect a function about the y-axis, we replace every x with $-x$. Similarly, to reflect a function about the x-axis, we replace every y with $-y$. To reflect a function about the line $y = x$, we replace x with y and y with x, which is how we determine the inverse.



b) Will the line $y = -x$ be an axis of symmetry for a function and its inverse?
Solution: No it will not.

5. a) Given $f^{-1}(x) = -2x + 4$, determine $f(x)$.
Solution:

$$\begin{aligned} f^{-1}(x) : y &= -2x + 4 \\ f(x) : x &= -2y + 4 \\ \therefore 2y &= -x + 4 \\ y &= -\frac{1}{2}x + 2 \\ \therefore f(x) &= -\frac{1}{2}x + 2 \end{aligned}$$

b) Calculate the intercepts of $f(x)$ and $f^{-1}(x)$.
Solution:
Consider $f(x)$:

$$\begin{aligned} \text{Let } y &= -\frac{1}{2}x + 2 \\ \text{Let } x = 0 : \quad y &= -\frac{1}{2}(0) + 2 \\ y &= 2 \\ \therefore y - &\text{intercept is } (0; 2) \end{aligned}$$

$$\begin{aligned} \text{Let } y = 0 : \quad 0 &= -\frac{1}{2}x + 2 \\ -2 &= -\frac{1}{2}x \\ x &= 4 \\ \therefore x - &\text{intercept is } (4; 0) \end{aligned}$$

Consider $f^{-1}(x)$:

$$\begin{aligned} \text{Let } y &= -2x + 4 \\ \text{Let } x = 0 : \quad y &= -2(0) + 4 \\ y &= 4 \\ \therefore y - &\text{intercept is } (0; 4) \end{aligned}$$

$$\begin{aligned} \text{Let } y = 0 : \quad 0 &= -2x + 4 \\ 2x &= 4 \\ x &= 2 \\ \therefore x - &\text{intercept is } (2; 0) \end{aligned}$$

Therefore the intercepts for $f(x)$ are $(4; 0)$ and $(0; 2)$ and the intercepts for $f^{-1}(x)$ are $(2; 0)$ and $(0; 4)$.

c) Determine the coordinates of T, the point of intersection of $f(x)$ and $f^{-1}(x)$.

Solution:

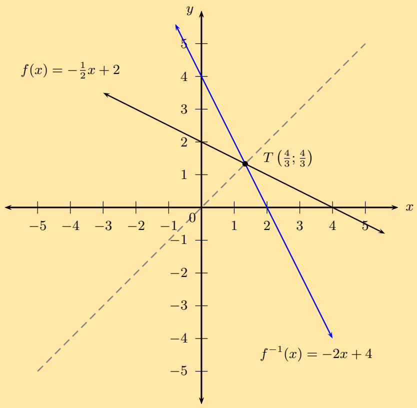
To find the point on intersection, we let $f(x) = f^{-1}(x)$:

$$\begin{aligned}
-\frac{1}{2}x + 2 &= -2x + 4 \\
-\frac{1}{2}x + 2x &= 4 - 2 \\
\frac{3}{2}x &= 2 \\
\therefore x &= \frac{4}{3} \\
\text{And } y &= -2\left(\frac{4}{3}\right) + 4 \\
&= -\frac{8}{3} + 4 \\
&= \frac{4}{3}
\end{aligned}$$

This gives the point $T\left(\frac{4}{3}; \frac{4}{3}\right)$.

d) Sketch the graphs of f and f^{-1} on the same system of axes. Indicate the intercepts and point T on the graph.

Solution:



e) Is f^{-1} an increasing or decreasing function?

Solution:

Decreasing function. The function values decrease as x increases.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28B4 2a. 28B5 2b. 28B6 2c. 28B7 3. 28B8 4. 28B9
5. 28BB

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Inverse of the function $y = ax^2$

Exercise 2 – 4: Inverses - domain, range, intercepts, restrictions

1. Determine the inverse for each of the following functions:

   a) $y = \frac{3}{4}x^2$

   Solution:

$$\begin{aligned} y &= \frac{3}{4}x^2 \\ \text{Interchange } x \text{ and } y: \quad x &= \frac{3}{4}y^2 \\ \frac{4}{3}x &= y^2 \\ \therefore y &= \pm\sqrt{\frac{4}{3}x} \qquad (x \geq 0) \end{aligned}$$

   b) $4y - 8x^2 = 0$

   Solution:

$$\begin{aligned} \text{Interchange } x \text{ and } y: \quad 4x - 8y^2 &= 0 \\ 4x &= 8y^2 \\ \frac{1}{2}x &= y^2 \\ \therefore y &= \pm\sqrt{\frac{1}{2}x} \qquad (x \geq 0) \end{aligned}$$

   c) $x^2 + 5y = 0$

   Solution:

$$\begin{aligned} \text{Interchange } x \text{ and } y: \quad y^2 + 5x &= 0 \\ y^2 &= -5x \\ \therefore y &= \pm\sqrt{-5x} \qquad (x \leq 0) \end{aligned}$$

   d) $4y - 9 = (x + 3)(x - 3)$

   Solution:

$$\begin{aligned} 4y - 9 &= x^2 - 9 \\ 4y &= x^2 \\ \text{Interchange } x \text{ and } y: \quad 4x &= y^2 \\ \therefore y &= \pm\sqrt{4x} \qquad (x \geq 0) \end{aligned}$$

2. Given the function $g(x) = \frac{1}{2}x^2$ for $x \geq 0$.

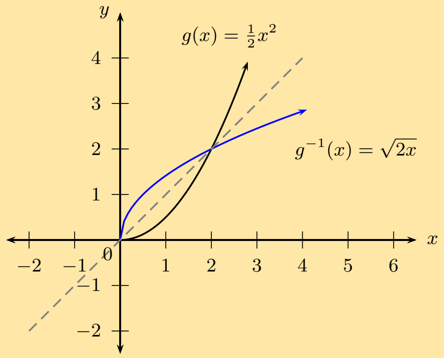
   a) Find the inverse of g.

Solution:

$$\text{Let } y = \frac{1}{2}x^2 \qquad (x \geq 0)$$
$$\text{Interchange } x \text{ and } y: \quad x = \frac{1}{2}y^2 \qquad (y \geq 0)$$
$$2x = y^2$$
$$y = \sqrt{2x} \qquad (x \geq 0, y \geq 0)$$

$$\therefore g^{-1}(x) = \sqrt{2x} \qquad (x \geq 0)$$

b) Draw g and g^{-1} on the same set of axes.
Solution:



c) Is g^{-1} a function? Explain your answer.
Solution:
Yes. It passes the vertical line test and is a one-to-one relation.

d) State the domain and range for g and g^{-1}.
Solution:

$$g: \quad \text{domain } x \geq 0 \quad \text{range } y \geq 0$$
$$g^{-1}: \quad \text{domain } x \geq 0 \quad \text{range } y \geq 0$$

e) Determine the coordinates of the point(s) of intersection of the function and its inverse.
Solution:
To find the points of intersection, we equate $g(x)$ and $g^{-1}(x)$:

$$\frac{1}{2}x^2 = \sqrt{2x}$$
$$\left(\frac{1}{2}x^2\right)^2 = \left(\sqrt{2x}\right)^2$$
$$\frac{1}{4}x^4 = 2x$$
$$x^4 = 8x$$
$$x^4 - 8x = 0$$
$$x(x^3 - 8) = 0$$
$$x(x-2)(x^2+2x+4) = 0$$
$$\therefore x = 0 \text{ or } x = 2 \text{ or } x^2 + 2x + 4 = 0$$

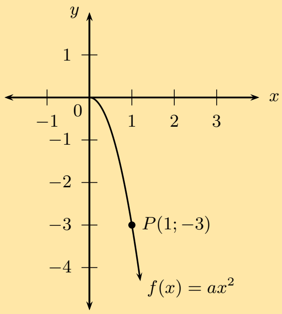
$$\text{If } x = 0$$
$$y = 0$$
$$\text{If } x = 2$$
$$y = \frac{1}{2}(2)^2$$
$$= 2$$

$$\text{If } x^2 + 2x + 4 = 0$$

$$\begin{aligned} \text{Use the quadratic formula } x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{-12}}{2} \\ &= \text{no real solution} \end{aligned}$$

Therefore, we get the points $(0; 0)$ or $(2; 2)$.

3. Given the graph of the parabola $f(x) = ax^2$ with $x \geq 0$ and passing through the point $P(1; -3)$.



a) Determine the equation of the parabola.
Solution:

$$\begin{aligned} \text{Let } y &= ax^2 \\ \text{Substitute } P(1; -3): \quad -3 &= a(1)^2 \\ \therefore a &= -3 \\ \therefore f(x) &= -3x^2 \quad (x \geq 0) \end{aligned}$$

b) State the domain and range of f.
Solution:
$f:$ domain $\{x : x \geq 0\}$ range $\{y : y \leq 0\}$

c) Give the coordinates of the point on f^{-1} that is symmetrical to the point P about the line $y = x$.
Solution: $(-3; 1)$

d) Determine the equation of f^{-1}.
Solution:

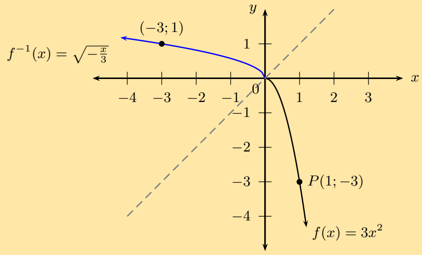
$$\begin{aligned} \text{Let } y &= -3x^2 \qquad (x \geq 0) \\ \text{Interchange } x \text{ and } y: \quad x &= -3y^2 \qquad (y \geq 0) \\ -\frac{1}{3}x &= y^2 \\ y &= \sqrt{-\frac{1}{3}x} \qquad (x \leq 0, y \geq 0) \end{aligned}$$

$$\therefore f^{-1}(x) = \sqrt{-\frac{1}{3}x} \qquad (x \leq 0)$$

e) State the domain and range of f^{-1}.
Solution:
$f^{-1}:$ domain $\{x : x \leq 0\}$ range $\{y : y \geq 0\}$

f) Draw a graph of f^{-1}.

Solution:



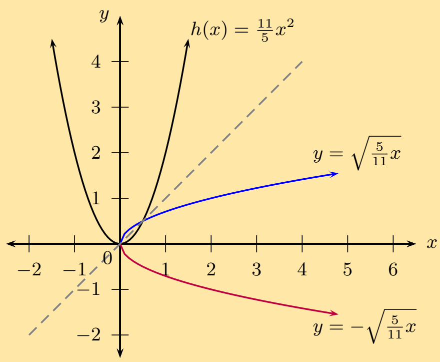
4. a) Determine the inverse of $h(x) = \frac{11}{5}x^2$.

Solution:

$$\begin{aligned} \text{Let } y &= \frac{11}{5}x^2 \\ \text{Interchange } x \text{ and } y: \quad x &= \frac{11}{5}y^2 \\ \frac{5}{11} &= y^2 \\ \therefore y &= \pm\sqrt{\frac{5}{11}x} \qquad (x \geq 0) \end{aligned}$$

b) Sketch both graphs on the same system of axes.

Solution:

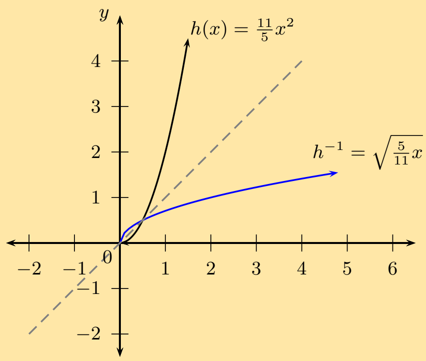


c) Restrict the domain of h so that the inverse is a function.

Solution:

Option 1: Restrict the domain of h to $x \geq 0$ so that the inverse will also be a function (h^{-1}). The restriction $x \geq 0$ on the domain of h will restrict the range of h^{-1} such that $y \geq 0$.

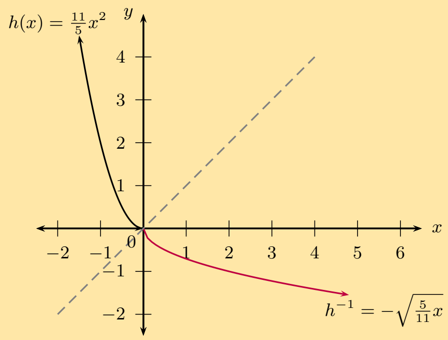
h :	domain $x \geq 0$	range $y \geq 0$
h^{-1} :	domain $x \geq 0$	range $y \geq 0$



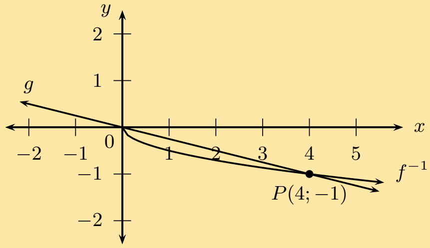
or

Option 2: Restrict the domain of h to $x \leq 0$ so that the inverse will also be a function h^{-1}). The restriction $x \leq 0$ on the domain of h will restrict the range of h^{-1} such that $y \leq 0$.

$$\begin{aligned} h: &\quad \text{domain } x \leq 0 \quad \text{range } y \geq 0 \\ h^{-1}: &\quad \text{domain } x \geq 0 \quad \text{range } y \leq 0 \end{aligned}$$



5. The diagram shows the graph of $g(x) = mx + c$ and $f^{-1}(x) = a\sqrt{x}, \ (x \geq 0)$. Both graphs pass through the point $P(4;-1)$.



a) Determine the values of a, c and m.

Solution:

From the graph we see that the straight line passes through the origin, therefore $c = 0$.

$$\begin{aligned} g(x) &= mx \\ \text{Substitute } P(4;-1): \quad -1 &= 4m \\ \therefore m &= -\frac{1}{4} \\ \therefore g(x) &= -\frac{1}{4}x \end{aligned}$$

$$\begin{aligned} f^{-1}(x) &= a\sqrt{x} \\ \text{Substitute } P(4;-1): \quad -1 &= a\sqrt{4} \\ -1 &= 2a \\ \therefore a &= -\frac{1}{2} \\ \therefore f^{-1}(x) &= -\frac{1}{2}\sqrt{x} \end{aligned}$$

b) Give the domain and range of f^{-1} and g.

Solution:

$$\begin{aligned} g: &\quad \text{domain: } x \in \mathbb{R} \quad \text{range: } y \in \mathbb{R} \\ f^{-1}: &\quad \text{domain: } x \geq 0 \quad \text{range: } y \leq 0 \end{aligned}$$

c) For which values of x is $g(x) < f(x)$?

Solution:

$x > 4$

d) Determine f.

Solution:

$$\begin{aligned} \text{Let } y &= -\frac{1}{2}\sqrt{x} \\ \text{Interchange } x \text{ and } y: \quad x &= -\frac{1}{2}\sqrt{y} \qquad (y \geq 0) \\ -2x &= \sqrt{y} \\ \therefore y &= 4x^2 \qquad (y \geq 0) \end{aligned}$$

e) Determine the coordinates of the point(s) of intersection of g and f intersect.

Solution:

To determine the coordinates of the point(s) of intersection, we equate g and f:

$$\begin{aligned} -\frac{1}{4}x &= 4x^2 \\ 0 &= 4x^2 + \frac{1}{4}x \\ 0 &= x\left(4x + \frac{1}{4}\right) \\ \therefore x = 0 \text{ or } 4x + \frac{1}{4} &= 0 \\ \text{If } x = 0: \quad y &= 0 \\ \text{If } 4x + \frac{1}{4} &= 0: \\ 4x &= -\frac{1}{4} \\ x &= -\frac{1}{16} \\ \text{If } x = -\frac{1}{16}: \quad y &= -\frac{1}{4}\left(-\frac{1}{16}\right) \\ &= \frac{1}{64} \end{aligned}$$

Therefore, the two graphs intersect at $(0;0)$ and $\left(-\frac{1}{16};\frac{1}{64}\right)$.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28BC 1b. 28BD 1c. 28BF 1d. 28BG 2. 28BH 3. 28BJ
4. 28BK 5. 28BM

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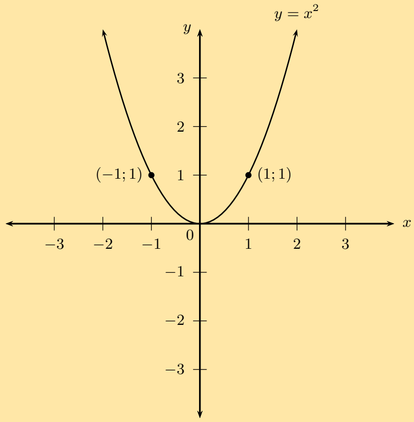
Exercise 2 – 5: Inverses - average gradient, increasing and decreasing functions

1. a) Sketch the graph of $y = x^2$ and label a point other than the origin on the graph.

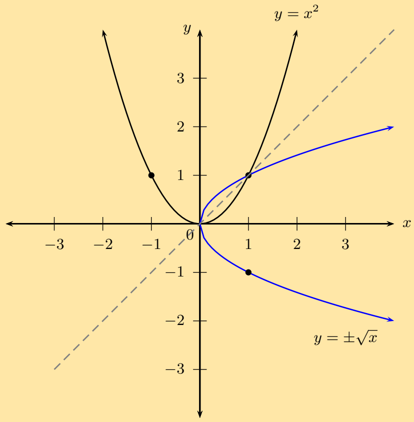
Solution:



b) Find the equation of the inverse of $y = x^2$.
Solution:

$$
\begin{aligned}
y &= x^2 \\
\text{Inverse: } x &= y^2 \\
\therefore y &= \pm\sqrt{x} \qquad (x \geq 0)
\end{aligned}
$$

c) Sketch the graph of the inverse on the same system of axes.
Solution:



d) Is the inverse a function? Explain your answer.
Solution:
No. For certain values of x, the inverse cuts a vertical line in two places. Therefore, it is not function.

e) $P(2;4)$ is a point on $y = x^2$. Determine the coordinates of Q, the point on the graph of the inverse which is symmetrical to P about the line $y = x$.
Solution:
$Q(4;2)$

f) Determine the average gradient between:
   i. the origin and P;
   ii. the origin and Q.

Interpret the answers.
Solution:
   i.

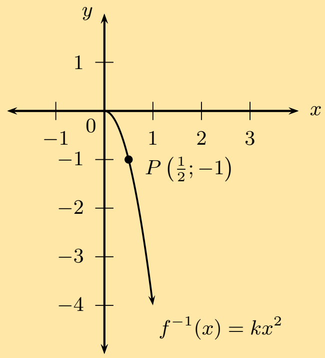
$$
\begin{aligned}
\text{Average gradient: } &= \frac{y_P - y_O}{x_P - x_O} \\
&= \frac{4-0}{2-0} \\
&= 2
\end{aligned}
$$

ii.

$$\begin{aligned} \text{Average gradient: } &= \frac{y_Q - y_O}{x_Q - x_O} \\ &= \frac{2-0}{4-0} \\ &= \frac{1}{2} \end{aligned}$$

Average gradient$_{OP} = 2$ and average gradient$_{OQ} = \frac{1}{2}$.
Both gradients are positive, and they are also reciprocals of each other.

2. Given the function $f^{-1}(x) = kx^2$, $x \geq 0$, which passes through the point $P\left(\frac{1}{2};-1\right)$.



a) Find the value of k.
Solution:

$$\begin{aligned} f^{-1}(x) &= kx^2 \\ \text{Substitute } \left(\frac{1}{2};-1\right) \quad -1 &= k\left(\frac{1}{2}\right)^2 \\ -1 &= k\left(\frac{1}{4}\right) \\ -4 &= k \\ \therefore f^{-1}(x) &= -4x^2 \end{aligned}$$

b) State the domain and range of f^{-1}.
Solution:

$$\begin{aligned} \text{Domain: } &\{x : x \geq 0, x \in \mathbb{R}\} \\ \text{Range: } &\{y : y \leq 0, y \in \mathbb{R}\} \end{aligned}$$

c) Find the equation of f.
Solution:

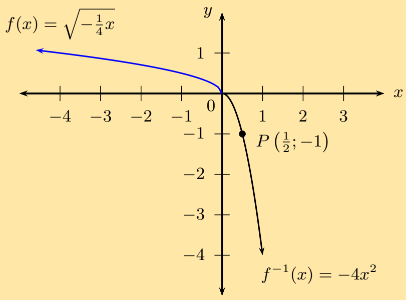
$$\begin{aligned} f^{-1}: \quad y &= -4x^2 \quad (x \geq 0) \\ f: \quad x &= -4y^2 \quad (y \geq 0) \\ -\frac{1}{4}x &= y^2 \\ \therefore y &= \sqrt{-\frac{1}{4}x} \quad (x \leq 0) \end{aligned}$$

d) State the domain and range of f.
Solution:

$$\begin{aligned} \text{Domain: } &\{x : x \leq 0, x \in \mathbb{R}\} \\ \text{Range: } &\{y : y \geq 0, y \in \mathbb{R}\} \end{aligned}$$

e) Sketch the graphs of f and f^{-1} on the same system of axes.

Solution:



f) Is f an increasing or decreasing function?

Solution:

Decreasing function: as the value of x increases, the function value decreases.

3. Given: $g(x) = \frac{5}{2}x^2, \ \ x \geq 0$.

a) Find $g^{-1}(x)$.

Solution:

$$\begin{aligned}
\text{Let } y &= \frac{5}{2}x^2 \qquad (x \geq 0) \\
\text{Interchange } x \text{ and } y: \quad x &= \frac{5}{2}y^2 \qquad (y \geq 0) \\
\frac{2}{5}x &= y^2 \\
y &= \sqrt{\frac{2}{5}x} \qquad (x \geq 0, y \geq 0) \\
\therefore g^{-1}(x) &= \sqrt{\frac{2}{5}x} \qquad (x \geq 0)
\end{aligned}$$

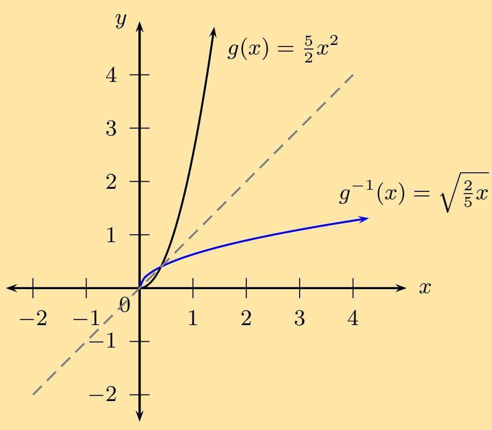
b) Calculate the point(s) where g and g^{-1} intersect.

Solution:

$$\frac{5}{2}x^2 = \sqrt{\frac{2}{5}x}$$
$$\left(\frac{5}{2}x^2\right)^2 = \left(\sqrt{\frac{2}{5}x}\right)^2$$
$$\frac{25}{4}x^4 = \frac{2}{5}x$$
$$\frac{25}{4}x^4 - \frac{2}{5}x = 0$$
$$125x^4 - 8x = 0$$
$$x(125x^3 - 8) = 0$$
$$x(5x-2)(25x^2+10x+4) = 0$$
$$\therefore x = 0 \quad \text{or} \quad 5x - 2 = 0 \quad \text{or} \quad 25x^2 + 10x + 4 = 0$$
$$\text{If } x = 0, \quad y = 0$$
$$\text{If } x = \frac{2}{5}, \quad y = \frac{5}{2}\left(\frac{2}{5}\right)^2$$
$$\therefore y = \frac{2}{5}$$
$$\text{If } 25x^2 + 10x + 4 = 0$$
$$\text{Use quadratic formula:} \quad x = \frac{-10 \pm \sqrt{100 - 4(25)(4)}}{2(25)}$$
$$= \frac{-10 \pm \sqrt{-300}}{50}$$
$$\therefore \text{no real solution}$$

Therefore, the points of intersection are $(0;0)$ and $\left(\frac{2}{5};\frac{2}{5}\right)$.

c) Sketch g and g^{-1} on the same set of axes.
Solution:



d) Use the sketch to determine if g and g^{-1} are increasing or decreasing functions.
Solution:

$$g: \text{ as } x \text{ increases, } y \text{ also increases, } \therefore g \text{ is an increasing function.}$$
$$g^{-1}: \text{ as } x \text{ increases, } y \text{ also increases, } \therefore g^{-1} \text{ is an increasing function.}$$

e) Calculate the average gradient of g^{-1} between the two points of intersection.
Solution:

$$\text{Average gradient: } = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{\frac{2}{5} - 0}{\frac{2}{5} - 0}$$
$$= 1$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28BN 2. 28BP 3. 28BQ

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2.6 Exponential functions

Inverse of the function $y = b^x$

Exercise 2 – 6: Finding the inverse of $y = b^x$

1. Write the following in logarithmic form:

   a) $16 = 2^4$

   Solution:

   $4 = \log_2 16$

   b) $3^{-5} = \frac{1}{243}$

   Solution:

   $-5 = \log_3 \left(\frac{1}{243}\right)$

   c) $(1{,}7)^3 = 4{,}913$

   Solution:

   $3 = \log_{1{,}7}(4{,}913)$

   d) $y = 2^x$

   Solution:

   $x = \log_2 y$

   e) $q = 4^5$

   Solution:

   $\log_4 q = 5.$

   f) $4 = y^g$

   Solution:

   $\log_y 4 = g.$

   g) $9 = (x-4)^p$

   Solution:

   $\log_{(x-4)} 9 = p.$

   h) $3 = m^{(a+4)}$

   Solution:

   $\log_m 3 = a + 4.$

2. Express each of the following logarithms in words and then write in exponential form:

   a) $\log_2 32 = 5$

   Solution:

   The logarithm of 32 to base 2 is equal to 5.

   $2^5 = 32$

   b) $\log \frac{1}{1000} = -3$

   Solution:

   The logarithm of $\frac{1}{1000}$ to base 10 is equal to -3.

   $10^{-3} = \frac{1}{1000}$

   c) $\log 0{,}1 = -1$

Solution:

The logarithm of $0{,}1$ to base 10 is equal to -1.

$10^{-1} = 0{,}1$

d) $\log_d c = b$

Solution:

The logarithm of c to base d is equal to b.

$d^b = c$

e) $\log_5 1 = 0$

Solution:

The logarithm of 1 to base 5 is equal to 0.

$5^0 = 1$

f) $\log_3 \frac{1}{81} = -4$

Solution:

The logarithm of $\frac{1}{81}$ to base 3 is equal to -4.

$3^{-4} = \frac{1}{81}$

g) $\log 100$

Solution:

$$\begin{aligned} \text{Let } \log 100 &= m \\ 10^m &= 100 \\ &= 10^2 \\ \therefore m &= 2 \\ \therefore \log 100 &= 2 \end{aligned}$$

The logarithm of 100 to base 10 is 2.

h) $\log_{\frac{1}{2}} 16$

Solution:

$$\begin{aligned} \text{Let } \log_{\frac{1}{2}} 16 &= y \\ \therefore \left(\frac{1}{2}\right)^y &= 16 \\ \therefore 2^{-y} &= 2^4 \\ -y &= 4 \\ \therefore y &= -4 \end{aligned}$$

The logarithm of 16 to base $\frac{1}{2}$ is -4.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28BS	1b. 28BT	1c. 28BV	1d. 28BW	1e. 28BX	1f. 28BY
1g. 28BZ	1h. 28C2	2a. 28C3	2b. 28C4	2c. 28C5	2d. 28C6
2e. 28C7	2f. 28C8	2g. 28C9	2h. 28CB		

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Exercise 2 – 7: Applying the logarithmic law: $\log_a x^b = b \log_a x$

Simplify the following:

1. $\log_8 10^{10}$

   Solution:

$$\log_8 10^{10} = 10 \log_8 10$$

2. $\log_{16} x^y$

   Solution:

$$\log_{16} x^y = y \log_{16} x$$

3. $\log_3 \sqrt{5}$

   Solution:

$$\log_3 \sqrt{5} = \frac{\log_3 5}{2}$$

4. $\log_z y^z$

   Solution:

$$\log_z y^z = z \log_z y$$

5. $\log_y \sqrt[x]{y}$

   Solution:

$$\begin{aligned} \log_y \sqrt[x]{y} &= \log_y y^{\frac{1}{x}} \\ &= \frac{1}{x} \log_y y \\ &= \frac{1}{x} \end{aligned}$$

6. $\log_p p^q$

   Solution:

$$\begin{aligned} \log_p p^q &= q \log_p p \\ &= q(1) \\ &= q \end{aligned}$$

7. $\log_2 \sqrt[4]{8}$

   Solution:

$$\begin{aligned} \log_2 \sqrt[4]{8} &= \log_2 8^{\frac{1}{4}} \\ &= \frac{1}{4} \log_2 2^3 \\ &= \frac{1}{4} \times 3 \log_2 2 \\ &= \frac{3}{4} \end{aligned}$$

8. $\log_5 \frac{1}{5}$

Solution:

$$\begin{aligned} \log_5 \frac{1}{5} &= \log_5 5^{-1} \\ &= (-1)\log_5 5 \\ &= (-1)(1) \\ &= -1 \end{aligned}$$

9. $\log_2 8^5$

Solution:

$$\begin{aligned} \log_2 8^5 &= 5\log_2 8 \\ &= 5\log_2 2^3 \\ &= 5 \times 3\log_2 2 \\ &= 15 \end{aligned}$$

10. $\log_4 16 \times \log_3 81$

Solution:

$$\begin{aligned} \log_4 16 \times \log_3 81 &= \log_4 4^2 \times \log_3 3^4 \\ &= (2)\log_4 4 \times (4)\log_3 3 \\ &= (2)(1) \times (4)(1) \\ &= 8 \end{aligned}$$

11. $(\log_5 25)^2$

Solution:

$$\begin{aligned} (\log_5 25)^2 &= \left(\log_5 5^2\right)^2 \\ &= (2\log_5 5)^2 \\ &= (2(1))^2 \\ &= 4 \end{aligned}$$

Alternative method:

$$\begin{aligned} (\log_5 25)^2 &= \log_5 \left(5^2\right)^2 \\ &= \log_5 5^4 \\ &= 4(1) \\ &= 4 \end{aligned}$$

12. $\log_2 0{,}125$

Solution:

$$\begin{aligned} \log_2 0{,}125 &= \log_2 \frac{125}{1000} \\ &= \log_2 \frac{1}{8} \\ &= \log_2 8^{-1} \\ &= (-1)\log_2 2^3 \\ &= (-1)(3)\log_2 2 \\ &= (-3)(1) \\ &= -3 \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28CD 2. 28CF 3. 28CG 4. 28CH 5. 28CJ 6. 28CK
7. 28CM 8. 28CN 9. 28CP 10. 28CQ 11. 28CR 12. 28CS

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Exercise 2 – 8: Applying the logarithmic law: $\log_a x = \frac{\log_b x}{\log_b a}$

1. Convert the following:

a) $\log_2 4$ to base 8

Solution:

$\log_2 4 = \frac{\log_8 4}{\log_8 2}$

b) $\log_{10} 14$ to base 2

Solution:

$\log_{10} 14 = \frac{\log_2 14}{\log_2 10}$

c) $\log 4\frac{1}{2}$ to base 2

Solution:

$$\begin{aligned} \log 4\frac{1}{2} &= \log \frac{9}{2} \\ &= \frac{\log_2 \frac{9}{2}}{\log_2 10} \\ &= \frac{\log_2 9 - \log_2 2}{\log_2 10} \\ &= \frac{\log_2 9 - 1}{\log_2 10} \end{aligned}$$

d) $\log_2 8$ to base 8

Solution:

$$\begin{aligned} \log_2 8 &= \frac{\log_8 8}{\log_8 2} \\ &= \frac{1}{\log_8 2} \end{aligned}$$

e) $\log_y x$ to base x

Solution:

$$\begin{aligned} \log_y x &= \frac{\log_x x}{\log_x y} \\ &= \frac{1}{\log_x y} \end{aligned}$$

$\therefore$ a logarithm is equal to the reciprocal of its inverse.

f) $\log_{10} 2x$ to base 2

Solution:

$$\begin{aligned} \log_{10} 2x &= \frac{\log_2 2x}{\log_2 10} \\ &= \frac{\log_2 2 + \log_2 x}{\log_2 10} \\ &= \frac{1 + \log_2 x}{\log_2 10} \end{aligned}$$

2. Simplify the following using a change of base:

a) $\log_2 10 \times \log_{10} 2$

Solution:

$$\begin{aligned} \log_2 10 \times \log_{10} 2 &= \frac{\log 10}{\log 2} \times \frac{\log 2}{\log 10} \\ &= 1 \end{aligned}$$

b) $\log_5 100$

Solution:

$$\begin{aligned} \log_5 100 &= \frac{\log 100}{\log 5} \\ &= \frac{\log 10^2}{\log 5} \\ &= \frac{2 \log 10}{\log 5} \\ &= \frac{2}{\log 5} \end{aligned}$$

3. If $\log 3 = 0{,}477$ and $\log 2 = 0{,}301$, determine (correct to 2 decimal places):

a) $\log_2 3$

Solution:

$$\begin{aligned} \log_2 3 &= \frac{\log 3}{\log 2} \\ &= \frac{0{,}477}{0{,}301} \\ &= 1{,}58 \end{aligned}$$

b) $\log_3 2000$

Solution:

$$\begin{aligned} \log_3 2000 &= \frac{\log 2000}{\log 3} \\ &= \frac{\log (2 \times 1000)}{\log 3} \\ &= \frac{\log 2 + \log 10^3}{\log 3} \\ &= \frac{0{,}301 + 3(1)}{0{,}477} \\ &= \frac{3{,}301}{0{,}477} \\ &= 6{,}92 \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28CV 1b. 28CW 1c. 28CX 1d. 28CY 1e. 28CZ 1f. 28D2
2a. 28D3 2b. 28D4 3a. 28D5 3b. 28D6

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Exercise 2 – 9: Logarithms using a calculator

1. Calculate the following (correct to three decimal places):

   a) $\log 3$
   Solution: $0{,}477$

   b) $\log 30$
   Solution: $1{,}477$

   c) $\log 300$
   Solution: $2{,}477$

   d) $\log 0.66$
   Solution: $-0{,}180$

   e) $\log \frac{1}{4}$
   Solution: $-0{,}602$

   f) $\log 852$
   Solution: $2{,}930$

   g) $\log(-6)$
   Solution: no value

   h) $\log_3 4$
   Solution: $1{,}262$

   i) $\log 0{,}01$
   Solution: -2

   j) $\log_2 15$
   Solution: $3{,}907$

   k) $\log_4 10$
   Solution: $1{,}661$

   l) $\log_{\frac{1}{2}} 6$
   Solution: $-2{,}585$

2. Use a calculator to determine the value of x (correct to two decimal places). Check your answer by changing to exponential form.

   a) $\log x = 0{,}6$
   Solution:

   $$x = 3{,}98$$
   $$\text{Check exponential form: } 10^{0{,}6} = 3{,}98$$

   b) $\log x = -2$
   Solution:

   $$x = 0{,}01$$
   $$\text{Check exponential form: } 10^{-2} = 0{,}01$$

   c) $\log x = 1{,}8$
   Solution:

   $$x = 63{,}10$$
   $$\text{Check exponential form: } 10^{1{,}8} = 63{,}10$$

d) $\log x = 5$

Solution:

$$x = 100\ 000$$

Check exponential form: $10^5 = 100\ 000$

e) $\log x = -0{,}5$

Solution:

$$x = 0{,}32$$

Check exponential form: $10^{-0{,}5} = 0{,}32$

f) $\log x = 0{,}076$

Solution:

$$x = 1{,}19$$

Check exponential form: $10^{0{,}076} = 1{,}19$

g) $\log x = \frac{2}{5}$

Solution:

$$x = 2{,}51$$

Check exponential form: $10^{\frac{2}{5}} = 2{,}51$

h) $\log x = -\frac{6}{5}$

Solution:

$$x = 0{,}06$$

Check exponential form: $10^{\left(-\frac{6}{5}\right)} = 0{,}06$

i) $\log_2 x = 0{,}25$

Solution:

$$\begin{aligned}
\log_2 x &= 0{,}25 \\
\therefore \frac{\log x}{\log 2} &= 0{,}25 \\
\therefore \log x &= 0{,}25 \times \log 2 \\
\therefore x &= 1{,}19
\end{aligned}$$

Check exponential form: $2^{0{,}25} = 1{,}19$

j) $\log_5 x = -0{,}1$

Solution:

$$\begin{aligned}
\log_5 x &= -0{,}1 \\
\therefore \frac{\log x}{\log 5} &= -0{,}1 \\
\therefore \log x &= -0{,}1 \times \log 5 \\
\therefore x &= 0{,}85
\end{aligned}$$

Check exponential form: $5^{(-0{,}10)} = 0{,}85$

k) $\log_{\frac{1}{4}} x = 2$

Solution:

$$\log_{\frac{1}{4}} x = 2$$
$$\therefore \frac{\log x}{\log \frac{1}{4}} = 2$$
$$\therefore \log x = 2 \times \log \frac{1}{4}$$
$$\therefore x = 0{,}06$$
$$\text{Check exponential form: } \left(\frac{1}{4}\right)^2 = 0{,}06$$

l) $\log_7 x = 0{,}3$

Solution:

$$\log_7 x = 0{,}3$$
$$\therefore \frac{\log x}{\log 7} = 0{,}3$$
$$\therefore \log x = 0{,}3 \times \log 7$$
$$\therefore x = 1{,}79$$
$$\text{Check exponential form: } 7^{0{,}3} = 1{,}79$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28D7	1b. 28D8	1c. 28D9	1d. 28DB	1e. 28DC	1f. 28DD
1g. 28DF	1h. 28DG	1i. 28DH	1j. 28DJ	1k. 28DK	1l. 28DM
2a. 28DN	2b. 28DP	2c. 28DQ	2d. 28DR	2e. 28DS	2f. 28DT
2g. 28DV	2h. 28DW	2i. 28DX	2j. 28DY	2k. 28DZ	2l. 28F2

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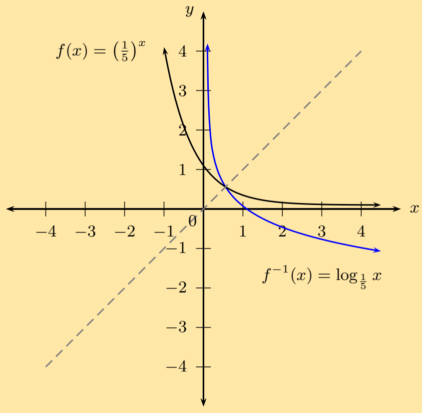
Exponential and logarithmic graphs

Exercise 2 – 10: Graphs and inverses of $y = \log_b x$

1. Given $f(x) = \left(\frac{1}{5}\right)^x$.

a) Sketch the graphs of f and f^{-1} on the same system of axes. Label both graphs clearly.

Solution:



b) State the intercept(s) for each graph.

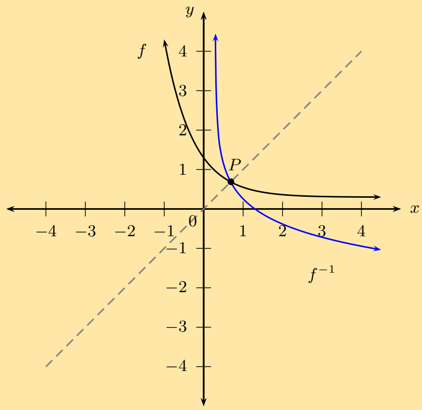
Solution:

$f:(0;1)$ and $f^{-1}:(1;0)$

c) Label P, the point of intersection of f and f^{-1}.

Solution:

Notice that the function and its inverse intersect at a point that lies on the line $y = x$.

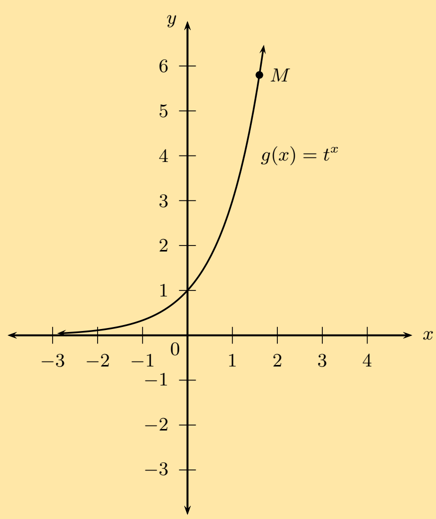


d) State the domain, range and asymptote(s) of each function.

Solution:

Function	Domain	Range	Asymptote
$f(x) = \left(\frac{1}{5}\right)^x$	$\{x : x \in \mathbb{R}\}$	$\{y : y > 0, y \in \mathbb{R}\}$	x-axis, $y = 0$
$f^{-1}(x) = \log_{\frac{1}{5}} x$	$\{x : x > 0, x \in \mathbb{R}\}$	$\{y : y \in \mathbb{R}\}$	y-axis, $x = 0$

2. Given $g(x) = t^x$ with $M\left(1\frac{3}{5}; 5\frac{4}{5}\right)$ a point on the graph of g.



a) Determine the value of t

Solution:

$$
\begin{aligned}
\text{Let } y &= t^x \\
5{,}8 &= t^{1{,}6} \\
\therefore t &= \sqrt[1{,}6]{5{,}8} \\
&= 3{,}000\ldots
\end{aligned}
$$

$$\therefore g(x) = 3^x$$

b) Find the inverse of g.

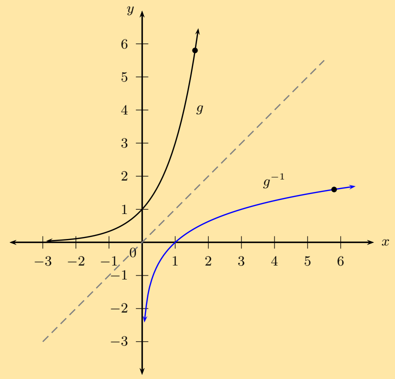
Solution:

$$\begin{aligned}
\text{Let } y &= 3^x && (y > 0) \\
\text{Interchange } x \text{ and } y: \quad x &= 3^y && (x > 0) \\
\frac{x}{2} &= y^2 \\
y &= \log_3 x
\end{aligned}$$

$$\therefore g^{-1}(x) = \log_3 x \qquad (x > 0)$$

c) Use symmetry about the line $y = x$ to sketch the graphs of g and g^{-1} on the same system of axes.

Solution:



d) Point N lies on the graph of g^{-1} and is symmetrical to point M about the line $y = x$. Determine the coordinates of N.

Solution:

$N(5{,}8; 1{,}6)$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28F4 2. 28F5

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Applications of logarithms

Exercise 2 – 11: Applications of logarithms

1. The population of Upington grows 6% every 3 years. How long will it take to triple in size?
   Give your answer in years and round to the nearest integer.
   Solution:
   Let the population at the start be
   $$P = x$$
   We want to know how many periods it will take to triple in size. This means that the final population is
   $$A = 3x$$
   Let n be the number of periods needed to grow to a size of A. Note that every period is 3 years long, therefore we must use $\frac{n}{3}$.

The growth rate is

$$i = 6\% = \frac{6}{100}$$

We use the formula for compound growth/interest:

$$\begin{aligned} A &= P(1+i)^n \\ 3x &= x\left(1+\frac{6}{100}\right)^{\frac{n}{3}} \\ 3 &= (1{,}06)^{\frac{n}{3}} \end{aligned}$$

We now use the definition of the logarithm to write the equation in terms of n and then evaluate with a calculator:

$$\begin{aligned} \frac{n}{3} &= \log_{(1{,}06)} 3 \\ &= \frac{\log 3}{\log 1{,}06} \\ &= 18{,}854\ldots \\ \therefore n &= 3 \times 18{,}854\ldots \\ &= 56{,}562\ldots \\ &\approx 57 \end{aligned}$$

It will take approximately 57 years for the population to triple in size.

2. An ant population of 36 ants doubles every month.

   a) Determine a formula that describes the growth of the population.

   Solution:

   Let the number of months be $M_0; M_1; M_2; \ldots$

M_0		M_1		M_2		$\ldots$
36	$\rightarrow$	72	$\rightarrow$	144	$\rightarrow$	$\ldots$
	$\times 2$		$\times 2$		$\times 2$	

   $$36 \times 2^0 \quad \rightarrow \quad 36 \times 2^1 \quad \rightarrow \quad 36 \times 2^2 \quad \rightarrow \quad \ldots$$

   $$\text{Growth} = 36 \times 2^n$$

   where n is the number of months.

   b) Calculate how long it will take for the ant population to reach a quarter of a million ants.

   Solution:

$$\begin{aligned} \text{Growth} &= 36 \times 2^n \\ \frac{1}{4} \times 1\,000\,000 &= 36 \times 2^n \\ 250\,000 &= 36 \times 2^n \\ \frac{250\,000}{36} &= 2^n \\ \therefore n &= \log_2\left(\frac{250\,000}{36}\right) \\ n &= \frac{\log\left(\frac{250\,000}{36}\right)}{\log 2} \\ \therefore n &= \frac{\log\left(\frac{250\,000}{36}\right)}{\log 2} \\ &= 12{,}7\ldots \end{aligned}$$

   There will be a quarter of a million ants after about 13 months.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28F7 2. 28F8

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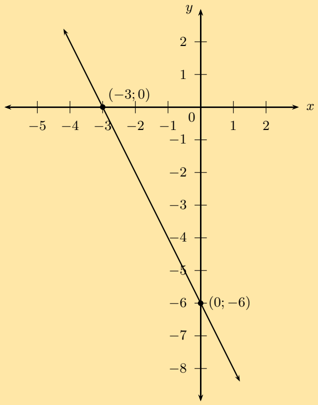
Exercise 2 – 12: End of chapter exercises

1. Given the straight line h with intercepts $(-3;0)$ and $(0;-6)$.



a) Determine the equation of h.

Solution:

$$\begin{aligned} \text{Gradient: } m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-6-0}{0-(-3)} \\ &= \frac{-6}{3} \\ &= -2 \\ \therefore h(x) &= -2x - 6 \end{aligned}$$

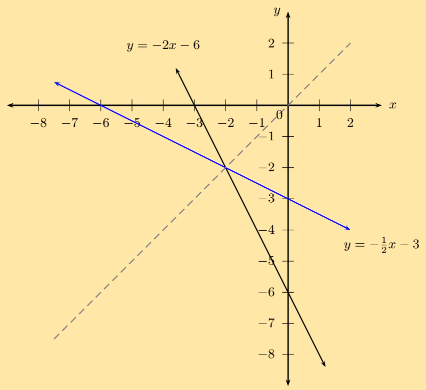
b) Find h^{-1}.

Solution:

$$\begin{aligned} \text{Let } y &= -2x - 6 \\ \text{Inverse: } x &= -2y - 6 \\ x + 6 &= -2y \\ -\frac{1}{2}(x+6) &= y \\ y &= -\frac{x}{2} - 3 \\ \therefore h^{-1}(x) &= -\frac{x}{2} - 3 \end{aligned}$$

c) Draw both graphs on the same system of axes.

Solution:



d) Calculate the coordinates of S, the point of intersection of h and h^{-1}.

Solution:

$$\begin{aligned} -\frac{x}{2} - 3 &= -2x - 6 \\ -x - 6 &= -4x - 12 \\ -x + 4x &= -12 + 6 \\ 3x &= -6 \\ \therefore x &= -2 \\ \text{If } x = -2, \quad y &= -2(-2) - 6 \\ y &= -2(-2) - 6 \\ &= -2 \end{aligned}$$

This gives the point $S(-2; -2)$

e) State the property regarding the point of intersection that will always be true for a function and its inverse.

Solution:

The value of the x-coordinate and the y-coordinate will always be the same since the point lies on the line $y = x$.

2. The inverse of a function is $f^{-1}(x) = 2x + 4$.

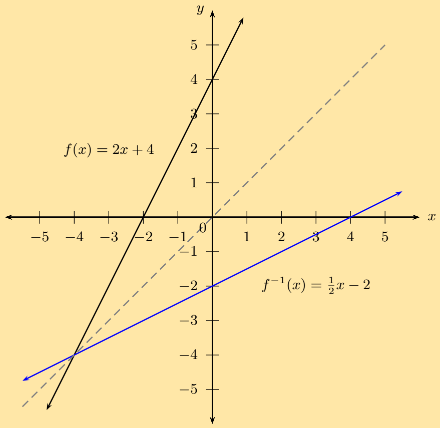
a) Determine f.

Solution:

$$\begin{aligned} \text{Let } y &= 2x + 4 \\ \text{Inverse: } x &= 2y + 4 \\ x - 4 &= 2y \\ \frac{1}{2}x - 2 &= y \\ \therefore f(x) &= \frac{1}{2}x - 2 \end{aligned}$$

b) Draw f and f^{-1} on the same set of axes. Label each graph clearly.

Solution:



c) Is f^{-1} an increasing or decreasing function? Explain your answer.

Solution:

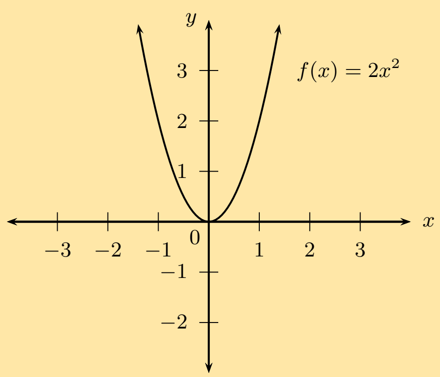
Increasing. As x increases, the function value increases. Alternative reason: gradient is positive, therefore function is increasing.

3. $f(x) = 2x^2$.

a) Draw the graph of f and state its domain and range.

Solution:

The domain is: $\{x : x \in \mathbb{R}\}$ and the range is: $\{y : y \geq 0, y \in \mathbb{R}\}$.



b) Determine the inverse and state its domain and range.

Solution:

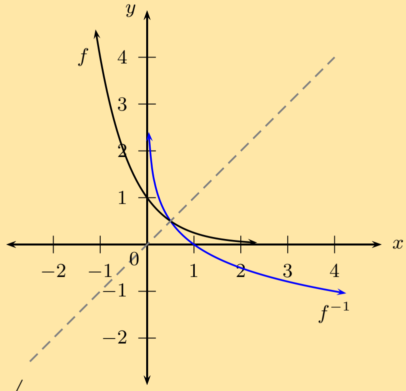
$$
\begin{aligned}
\text{Let } y &= 2x^2 \\
\text{Inverse: } x &= 2y^2 \\
\frac{1}{2}x &= y^2 \\
\pm\sqrt{\frac{1}{2}x} &= y \\
\therefore y &= \pm\sqrt{\frac{x}{2}} \qquad (x \geq 0)
\end{aligned}
$$

Domain: $\{x : x \geq 0, x \in \mathbb{R}\}$, Range: $\{y : y \in \mathbb{R}\}$.

4. Given the function $f(x) = \left(\frac{1}{4}\right)^x$.

a) Sketch the graphs of f and f^{-1} on the same system of axes.

Solution:



b) Determine if the point $\left(-\frac{1}{2};2\right)$ lies on the graph of f.
Solution:

$$\begin{aligned} f(x) &= \left(\frac{1}{4}\right)^{x} \\ \text{Substitute } \left(-\frac{1}{2};2\right): \quad f\left(-\frac{1}{2}\right) &= \left(\frac{1}{4}\right)^{-\frac{1}{2}} \\ &= 4^{\frac{1}{2}} \\ &= 2 \end{aligned}$$

Yes, the point $\left(-\frac{1}{2};2\right)$ does lie on f.

c) Write f^{-1} in the form $y = \ldots$
Solution:

$$\begin{aligned} f: \quad y &= \left(\frac{1}{4}\right)^{x} \\ f^{-1}: \quad x &= \left(\frac{1}{4}\right)^{y} \\ \therefore y &= \log_{\frac{1}{4}} x \end{aligned}$$

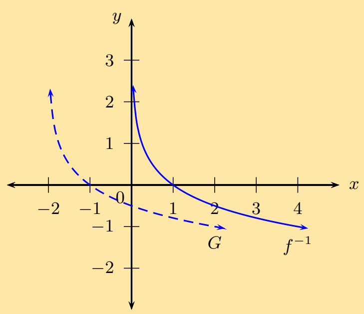
or

$$\begin{aligned} f: \quad y &= (4)^{-x} \\ f^{-1}: \quad x &= (4)^{-y} \\ -y &= \log_4 x \\ \therefore y &= -\log_4 x \end{aligned}$$

$y = \log_{\frac{1}{4}} x$ or $y = -\log_4 x$

d) If the graphs of f and f^{-1} intersect at $\left(\frac{1}{2};P\right)$, determine the value of P.
Solution:
$P = \frac{1}{2}$, since the point lies on the line $y = x$.

e) Give the equation of the new graph, G, if the graph of f^{-1} is shifted 2 units to the left.
Solution:



$G(x) = -\log_4(x+2)$ or $G(x) = \log_{\frac{1}{4}}(x+2)$

f) Give the asymptote(s) of G.

Solution:

Vertical asymptote: $x = -2$

5. Consider the function $h(x) = 3^x$.

a) Write down the inverse in the form $h^{-1}(x) = \ldots$

Solution:

$$\begin{aligned} \text{Let:}\quad & y = 3^x \\ \text{Inverse:}\ & x = 3^y \\ & y = \log_3 x \\ \therefore h^{-1}(x) &= \log_3 x \end{aligned}$$

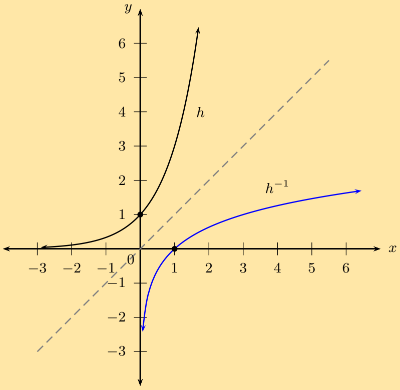
b) State the domain and range of h^{-1}.

Solution:

Domain: $\{x : x > 0, x \in \mathbb{R}\}$ and range: $\{y : y \in \mathbb{R}\}$.

c) Sketch the graphs of h and h^{-1} on the same system of axes, label all intercepts.

Solution:



d) For which values of x will $h^{-1}(x) < 0$?

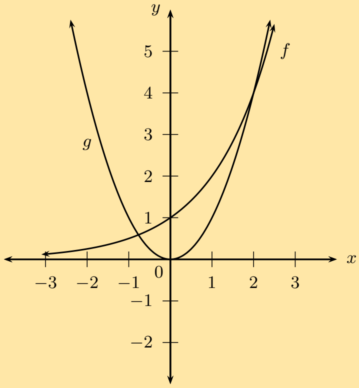
Solution:

$0 < x < 1$

6. Consider the functions $f(x) = 2^x$ and $g(x) = x^2$.

a) Sketch the graphs of f and g on the same system of axes.

Solution:



b) Determine whether or not f and g intersect at a point where $x = -1$.

Solution:

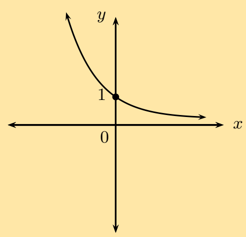
$$\begin{aligned} f(x) &= 2^x \\ f(-1) &= 2^{-1} \\ &= \frac{1}{2} \\ g(x) &= x^2 \\ g(-1) &= (-1)^2 \\ &= 1 \\ \therefore f(-1) &\neq g(-1) \end{aligned}$$

The graphs do not intersect at $x = -1$.

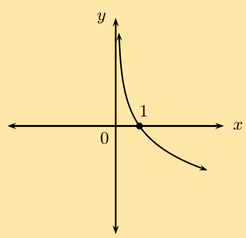
c) How many solutions does the equation $2^x = x^2$ have?

Solution:
Two

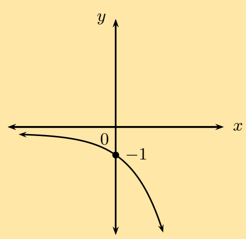
7. Below are three graphs and six equations. Write down the equation that best matches each of the graphs.



Graph 1



Graph 2

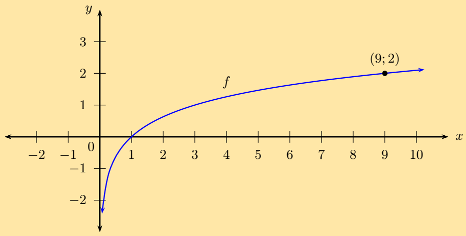


Graph 3

a) $y = \log_3 x$
b) $y = -\log_3 x$
c) $y = \log_{\frac{1}{3}} x$
d) $y = 3^x$
e) $y = 3^{-x}$
f) $y = -3^x$

Solution:
Graph 1: $y = 3^{-x}$
Graph 2: $y = -\log_3 x$ or $y = \log_{\frac{1}{3}} x$
Graph 3: $y = -3^x$

8. Given the graph of the function $f: \quad y = \log_b x$ passing through the point $(9; 2)$.



a) Show that $b = 3$.

Solution:

$$\begin{aligned} y &= \log_b x \\ 2 &= \log_b 9 \\ \therefore 9 &= b^2 \\ \therefore 3^2 &= b^2 \\ \therefore b &= 3 \end{aligned}$$

b) Determine the value of a if $(a; -1)$ lies on f.

Solution:

$$\begin{aligned} y &= \log_3 x \\ -1 &= \log_3 a \\ \therefore 3^{-1} &= a \\ \therefore a &= \frac{1}{3} \end{aligned}$$

c) Write down the new equation if f is shifted 2 units upwards.

Solution: $y = \log_3 x + 2$

d) Write down the new equation if f is shifted 1 units to the right.

Solution: $y = \log_3 (x - 1)$

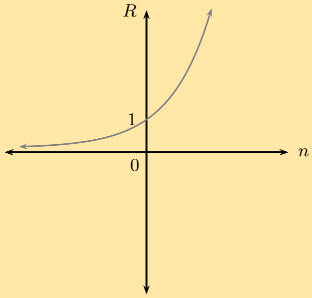
9. a) If the rhino population in South Africa starts to decrease at a rate of 7% per annum, determine how long it will take for the current rhino population to halve in size? Give your answer to the nearest integer.

Solution:

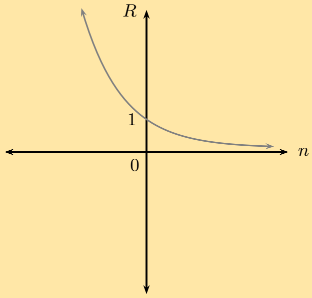
$$\begin{aligned} A &= P(1 - i)^n \\ \frac{1}{2} &= \left(1 - \frac{7}{100}\right)^n \\ 0{,}5 &= (0{,}93)^n \\ \log 0{,}5 &= \log (0{,}93)^n \\ \log 0{,}5 &= n \log (0{,}93) \\ \therefore n &= \frac{\log 0{,}5}{\log (0{,}93)} \\ &= 9{,}55\ldots \end{aligned}$$

It will take less than 10 years for the current rhino population to halve in size.

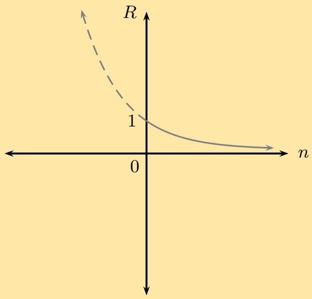
b) Which of the following graphs best illustrates the rhino population's decline? Motivate your answer.



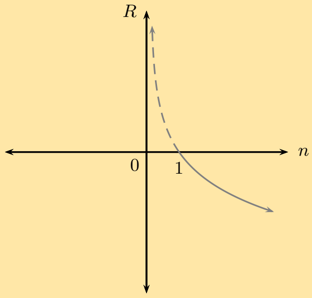
Graph A



Graph B



Graph C



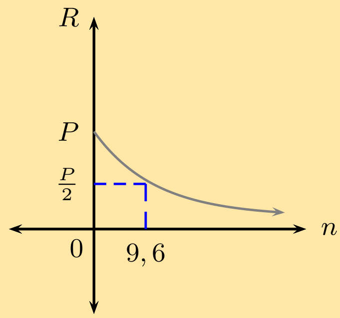
Graph D

Important note: the graphs above have been drawn as a continuous curve to show a trend. Rhino population numbers are discrete values and should be plotted points.

Solution:

Graph C

Currently $(n = 0)$ the rhino population is P. After $9{,}6$ years, it will have halved, $\frac{P}{2}$. Note: the line in the graph indicates the trend, rhino population numbers are discrete values and should be plotted points.



10. At 8 a.m. a local celebrity tweets about his new music album to 100 of his followers. Five minutes later, each of his followers retweet his message to two of their friends. Five minutes after that, each friend retweets the message to another two friends. Assume this process continues.

a) Determine a formula that describes this retweeting process.

Solution:

$100 \qquad 100 \times 2 \qquad 100 \times 2^2 \qquad 100 \times 2^3$

This is a geometric sequence: $r = 2$ and $a = 100$.

Therefore $T_n = 100 \times 2^{n-1}$.

b) Calculate how many retweets of the celebrity's message are sent an hour after his original tweet.

1 hour $= 60$ minutes $= 12 \times 5$, therefore $n = 12$.

Solution:

$$\begin{aligned} T_n &= 100 \times 2^{n-1} \\ T_{12} &= 100 \times 2^{11} \\ &= 204\ 800 \end{aligned}$$

$204\ 800$ retweets.

c) How long will it take for the total number of retweets to exceed 200 million?

Solution:

$$\begin{aligned} 200 \times 10^6 &= 2 \times 10^8 \\ S_n &= \frac{a(r^n - 1)}{r - 1} \\ \therefore \frac{100(2^n - 1)}{2 - 1} &> 2 \times 10^8 \\ 2^n &> \frac{2 \times 10^8}{100} + 1 \\ 2^n &> 2\ 000\ 001 \\ n &> \log_2 2\ 000\ 001 \qquad \text{(use definition)} \\ n &> \frac{\log 2\ 000\ 001}{\log 2} \qquad \text{(change of base)} \\ n &> 20{,}9\ldots \quad \text{5 minute periods} \end{aligned}$$

Therefore, $\frac{21}{12} = 1{,}75$ hours.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28FB	1b. 28FC	1c. 28FD	1d. 28FF	1e. 28FG	2. 28FH
3a. 28FJ	3b. 28FK	4. 28FM	5. 28FN	6. 28FP	7. 28FQ
8. 28FR	9. 28FS	10. 28FT			

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Exercise 2 – 13: Inverses (ENRICHMENT ONLY)

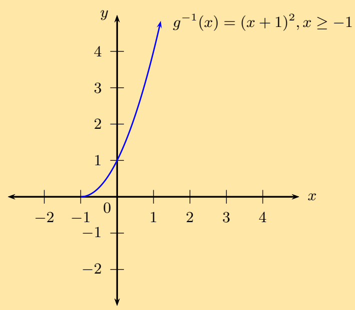
1. a) Given: $g\left(x\right) = -1 + \sqrt{x}$, find the inverse of $g\left(x\right)$ in the form $g^{-1}\left(x\right) = ...$
   Solution:

$$g(x) = -1 + \sqrt{x} \quad (x \geq 0)$$
$$\text{Let } y = -1 + \sqrt{x}$$
$$\text{Inverse: } x = -1 + \sqrt{y} \quad (y \geq 0)$$

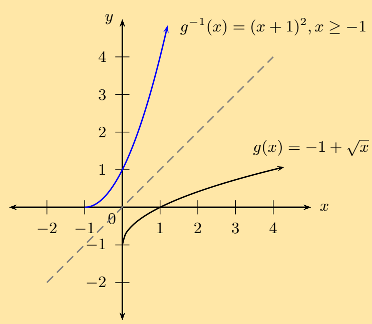
$$\sqrt{y} = x + 1 \qquad (x \geq -1)$$
$$\therefore y = (x+1)^2$$

$$\therefore g^{-1}(x) = (x+1)^2 \qquad (x \geq -1)$$

   b) Draw the graph of g^{-1}.
   Solution:



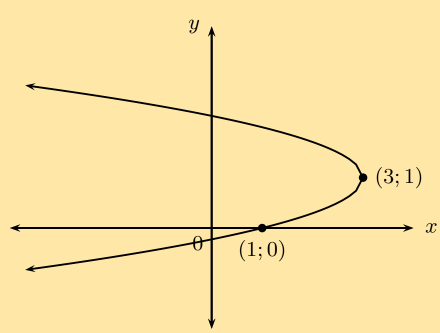
   c) Use symmetry to draw the graph of g on the same set of axes.
   Solution:



   d) Is g^{-1} a function?
   Solution:
   Yes. It passes vertical line test.

   e) Give the domain and range of g^{-1}.
   Solution:
   Domain: $\{x : x \geq -1, x \in \mathbb{R}\}$, Range: $\{y : y \geq 0, y \in \mathbb{R}\}$.

2. The graph of the inverse of f is shown below:



a) Find the equation of f, given that f is a parabola of the form $y = (x+p)^2 + q$.

Solution:

First use the information provided in the graph of the inverse:

$$\begin{aligned}\text{Turning point: } &(3;1)\\ x - \text{intercept: } &(1;0)\end{aligned}$$

To get the turning point and intercepts of the function, we invert the given coordinates. Now we can use those coordinates to find the equation of the function:

Now find the equation of the function:

$$\begin{aligned}\text{Turning point: } &(1;3)\\ x - \text{intercept: } &(0;1)\end{aligned}$$

$$\begin{aligned} & y = a(x-p)^2 + q\\ & y = a(x-1)^2 + 3\\ \text{Substitute } (0;1) \quad & 1 = a(0-1)^2 + 3\\ & a = -2\\ & \therefore y = -2(x-1)^2 + 3\end{aligned}$$

b) Will f have a maximum or a minimum value?

Solution:

Maximum value at $(1;3)$

c) State the domain, range and axis of symmetry of f.

Solution:

Domain: $\{x : x \in \mathbb{R}\}$ and range: $\{y : y \leq 3, y \in \mathbb{R}\}$, Axis of symmetry: $x = 1$.

3. Given: $k(x) = 2x^2 + 1$

a) If $(q;3)$ lies on k, determine the value(s) of q.

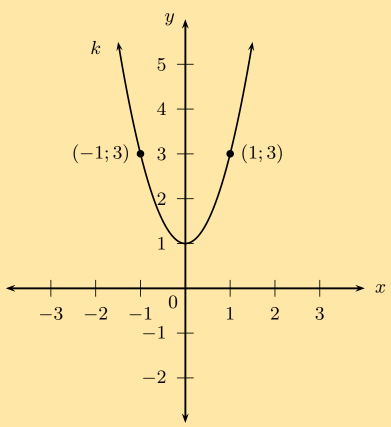
Solution:

$$\begin{aligned} & k(x) = 2x^2 + 1\\ \text{Substitute } (q;3) \quad & 3 = 2(q)^2 + 1\\ & 2 = 2(q)^2\\ & 1 = q^2\\ & \therefore q = \pm 1\end{aligned}$$

This gives the points $(-1;3)$ and $(1;3)$.

b) Sketch the graph of k, label the point(s) $(q;3)$ on the graph.

Solution:



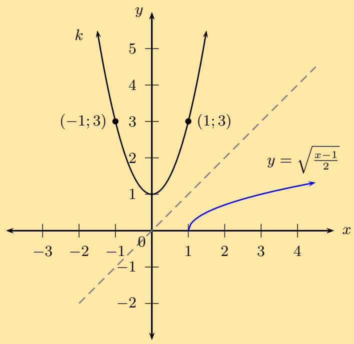
c) Find the equation of the inverse of k in the form $y = \ldots$

Solution:

$$\begin{aligned} k: \quad y &= 2x^2 + 1 \\ \text{Inverse:} \quad x &= 2y^2 + 1 \\ 2y^2 &= x - 1 \\ y^2 &= \frac{x-1}{2} \\ y &= \pm\sqrt{\frac{x-1}{2}} \qquad (x \geq 1) \end{aligned}$$

d) Sketch k and $y = \sqrt{\frac{x-1}{2}}$ on the same system of axes.

Solution:

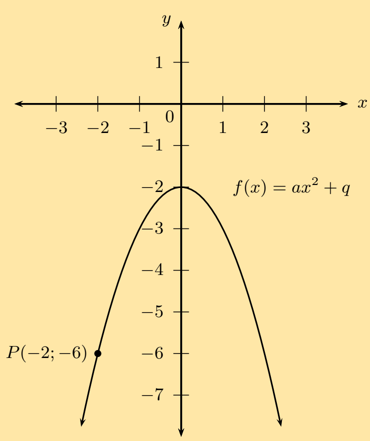


e) Determine the coordinates of the point on the graph of the inverse that is symmetrical to $(q;3)$ about the line $y = x$.

Solution:

$(3;1)$

4. The sketch shows the graph of a parabola $f(x) = ax^2 + q$ passing through the point $P(-2;-6)$.



a) Determine the equation of f.

Solution:

$$\begin{aligned} q &= -2 \\ y &= ax^2 - 2 \\ \text{Substitute } (-2;-6) \quad -6 &= a(-2)^2 - 2 \\ -6 + 2 &= 4a \\ -4 &= 4a \\ -1 &= a \\ \therefore f(x) &= -x^2 - 2 \end{aligned}$$

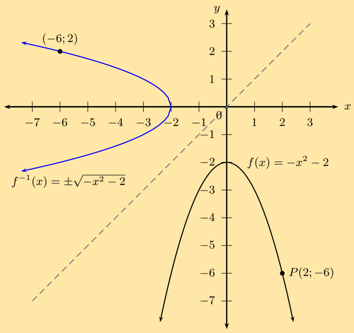
b) Determine and investigate the inverse.

Solution:

$$\begin{aligned} \text{Let } y &= -x^2 - 2 \qquad (y \leq -2) \\ \text{Interchange } x \text{ and } y: \quad x &= -y^2 - 2 \qquad (x \leq -2) \\ x + 2 &= -y^2 \\ -x - 2 &= y^2 \\ y &= \pm\sqrt{-x-2} \qquad (x \leq -2) \end{aligned}$$

c) Sketch the inverse and discuss the characteristics of the graph.

Solution:



The inverse is a not a function. The turning point of the inverse is $(-2;0)$ and x-intercept is $(-2;0)$.

Inverse : domain $\{x : x \leq -2, x \in \mathbb{R}\}$ range $\{y : y \in \mathbb{R}\}$

5. Given the function $H : y = x^2 - 9$.

a) Determine the algebraic formula for the inverse of H.

Solution:

$$\begin{aligned} \text{Let } y &= x^2 - 9 \qquad (y \geq -9) \\ \text{Interchange } x \text{ and } y: \quad x &= y^2 - 9 \qquad (x \geq -9) \\ x + 9 &= y^2 \\ y &= \pm\sqrt{x+9} \qquad (x \geq -9) \end{aligned}$$

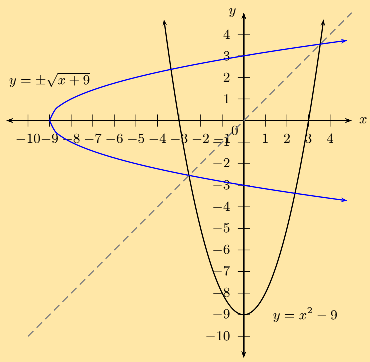
b) Draw graphs of H and its inverse on the same system of axes. Indicate intercepts and turning points.

Solution:

Determine the intercepts:

$$\begin{aligned} \text{Let } x = 0: \quad y &= (0)^2 - 9 \\ &= -9 \\ \text{Let } y = 0: \quad 0 &= x^2 - 9 \\ x^2 &= 9 \\ \therefore x &= \pm 3 \end{aligned}$$

The intercepts are $(0;-9)$ and $(-3;0)$, $(3;0)$.



c) Is the inverse a function? Give reasons.

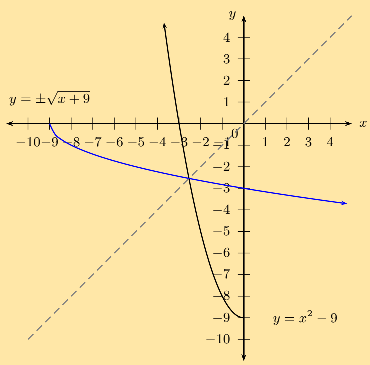
Solution:

No. The inverse does not pass the vertical line test. It is a one-to-many relation.

d) Show algebraically and graphically the effect of restricting the domain of H to $\{x : x \leq 0\}$.

Solution:

If the domain of H is restricted to $\{x : x \leq 0\}$, then the inverse is $H^{-1}(x) = -\sqrt{x^2+9} \quad (x \geq -9, y \leq 0)$.



The graph of H^{-1} cuts a vertical line only once at any one time and therefore passes the vertical line test.

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28FV 2. 28FW 3. 28FX 4a. 28FY 4b. 28FZ 4c. 28G2
5. 28G3

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2.8 Enrichment: more on logarithms

Laws of logarithms

Exercise 2 – 14: Applying logarithmic law: $\log_a xy = \log_a(x) + \log_a(y)$

1. Simplify the following, if possible:

a) $\log_8(10 \times 10)$

Solution:

$$\begin{aligned}\log_8 (10 \times 10) &= \log_8 10 + \log_8 10 \\ &= 2 \log_8 10\end{aligned}$$

b) $\log_2 14$

Solution:

$$\begin{aligned}\log_2 14 &= \log_2 (2 \times 7) \\ &= \log_2 2 + \log_2 7 \\ &= 1 + \log_2 7\end{aligned}$$

c) $\log_2 (8 \times 5)$

Solution:

$$\begin{aligned}\log_2 (8 \times 5) &= \log_2 (2 \times 2 \times 2 \times 5) \\ &= \log_2 2 + \log_2 2 + \log_2 2 + \log_2 5 \\ &= 3 \log_2 2 + \log_2 5 \\ &= 3(1) + \log_2 5\end{aligned}$$

d) $\log_{16} (x + y)$

Solution:

$\log_{16} (x + y)$ cannot be written as separate logarithms.

e) $\log_2 2xy$

Solution:

$$\begin{aligned}\log_2 2xy &= \log_2 (2 \times x \times y) \\ &= \log_2 2 + \log_2 x + \log_2 y \\ &= 1 + \log_2 x + \log_2 y\end{aligned}$$

f) $\log (5 + 2)$

Solution:

$\log (5 + 2) = \log 7$

Note: $\log (5 + 2) \neq \log 5 + \log 2$. Do not confuse this with applying the distributive law: $a(b + c) = ab + ac$.

2. Write the following as a single term, if possible:

a) $\log 15 + \log 2$

Solution:

$$\begin{aligned}\log 15 + \log 2 &= \log (15 \times 2) \\ &= \log 30\end{aligned}$$

b) $\log 1 + \log 5 + \log \frac{1}{5}$

Solution:

$$\begin{aligned}\log 1 + \log 5 + \log \frac{1}{5} &= \log \left(1 \times 5 \times \frac{1}{5}\right) \\ &= \log 1 \\ &= 0\end{aligned}$$

c) $1 + \log_3 4$

Solution:

$$\begin{aligned} 1 + \log_3 4 &= \log_3 3 + \log_3 4 \\ &= \log_3 (3 \times 4) \\ &= \log_3 12 \end{aligned}$$

d) $(\log x)(\log y) + \log x$

Solution:

$$\begin{aligned} (\log x)(\log y) + \log x &= (\log x)[\log y + 1] \\ &= (\log x)(\log y + \log 10) \\ &= (\log x)(\log 10y) \end{aligned}$$

e) $\log 7 \times \log 2$

Solution:

$$\begin{aligned} \log 7 \times \log 2 &= \log 7 \times \log 2 \\ &\text{cannot be simplified further} \\ \text{Note:} \quad \log 7 \times \log 2 &\neq \log(7+2) \end{aligned}$$

f) $\log_2 7 + \log_3 2$

Solution:

This cannot be written as one term because the bases are not the same.

g) $\log_a p + \log_a q$

Solution:

$$\begin{aligned} \log_a p + \log_a q &= \log_a (p \times q) \\ &= \log_a pq \end{aligned}$$

h) $\log_a p \times \log_a q$

Solution:

This is already a single term: $(\log_a p)(\log_a q)$

3. Simplify the following:

a) $\log x + \log y + \log z$

Solution:

$$\log_x + \log y + \log z = \log xyz$$

b) $\log ab + \log bc + \log cd$

Solution:

$$\log ab + \log bc + \log cd = \log ab^2c^2d$$

c) $\log 125 + \log 2 + \log 8$

Solution:

$$\begin{aligned} \log 125 + \log 2 + \log 8 &= \log(125 \times 2 \times 8) \\ &= \log 2000 \\ &= \log(2 \times 10 \times 10 \times 10) \\ &= \log 2 + \log 10 + \log 10 + \log 10 \\ &= \log 2 + 1 + 1 + 1 \\ &= \log 2 + 3 \end{aligned}$$

d) $\log_4 \frac{3}{8} + \log_4 \frac{10}{3} + \log_4 \frac{16}{5}$

Solution:

$$\begin{aligned}\log_4 \frac{3}{8} + \log_4 \frac{10}{3} + \log_4 \frac{16}{5} &= \log_4 \left(\frac{3}{8} \times \frac{10}{3} \times \frac{16}{5}\right)\\ &= \log_4 4\\ &= 1\end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28G4	1b. 28G5	1c. 28G6	1d. 28G7	1e. 28G8	1f. 28G9
2a. 28GB	2b. 28GC	2c. 28GD	2d. 28GF	2e. 28GG	2f. 28GH
2g. 28GJ	2h. 28GK	3a. 28GM	3b. 28GN	3c. 28GP	3d. 28GQ

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Exercise 2 – 15: Applying logarithmic law: $\log_a \frac{x}{y} = \log_a x - \log_a y$

1. Expand and simplify the following:

a) $\log \frac{100}{3}$
Solution:

$$\begin{aligned}\log \frac{100}{3} &= \log 100 - \log 3\\ &= \log (10 \times 10) - \log 3\\ &= \log 10 + \log 10 - \log 3\\ &= 1 + 1 - \log 3\\ &= 2 - \log 3\end{aligned}$$

b) $\log_2 7\frac{1}{2}$
Solution:

$$\begin{aligned}\log_2 7\frac{1}{2} &= \log_2 \frac{15}{2}\\ &= \log_2 15 - \log_2 2\\ &= \log_2 15 - 1\end{aligned}$$

c) $\log_{16} \frac{x}{y}$
Solution:

$$\log_{16} \frac{x}{y} = \log_{16} x - \log_{16} y$$

d) $\log_{16} (x - y)$
Solution:
This cannot be simplified.

e) $\log_5 \frac{5}{8}$
Solution:

$$\begin{aligned}\log_5 \frac{5}{8} &= \log_5 5 - \log_5 8\\ &= 1 - \log_5 8\end{aligned}$$

f) $\log_x \frac{y}{r}$

Solution:

$$\log_x \frac{y}{r} = \log_x y - \log_x r$$

2. Write the following as a single term:

   a) $\log 10 - \log 50$
   Solution:

$$\begin{aligned} \log 10 - \log 50 &= \log \frac{10}{50} \\ &= \log \frac{1}{5} \\ &= \log 5^{-1} \\ &= -\log 5 \end{aligned}$$

   b) $\log_3 36 - \log_3 4$
   Solution:

$$\begin{aligned} \log_3 36 - \log_3 4 &= \log_3 \frac{36}{4} \\ &= \log_3 9 \\ &= \log_3 (3 \times 3) \\ &= \log_3 3 + \log_3 3 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

   c) $\log_a p - \log_a q$
   Solution:

$$\log_a p - \log_a q = \log_a \frac{p}{q}$$

   d) $\log_a (p - q)$
   Solution:
   This cannot be simplified.

   e) $\log 15 - \log_2 5$
   Solution:
   This cannot be simplified because the bases are not the same.

   f) $\log 15 - \log 5$
   Solution:

$$\begin{aligned} \log 15 - \log 5 &= \log \frac{15}{5} \\ &= \log 3 \end{aligned}$$

3. Simplify the following:

   a) $\log 450 - \log 9 - \log 5$
   Solution:

$$\begin{aligned} \log 450 - \log 9 - \log 5 &= \log \left(\frac{450}{9 \times 5} \right) \\ &= \log 10 \\ &= 1 \end{aligned}$$

Alternative method:

$$\begin{aligned} \log 450 - \log 9 - \log 5 &= \log \frac{450}{9} - \log 5 \\ &= \log 50 - \log 5 \\ &= \log \frac{50}{5} \\ &= \log 10 \\ &= 1 \end{aligned}$$

b) $\log\frac{4}{5} - \log\frac{3}{25} - \log\frac{1}{15}$

Solution:

$$\begin{aligned}
\log\frac{4}{5} - \log\frac{3}{25} - \log\frac{1}{15} &= \log\left(\frac{\frac{4}{5}}{\frac{3}{25}\times\frac{1}{15}}\right) \\
&= \log\left(\frac{\frac{4}{5}}{\frac{1}{125}}\right) \\
&= \log 100 \\
&= 2
\end{aligned}$$

Alternative method:

$$\begin{aligned}
\log\frac{4}{5} - \log\frac{3}{25} - \log\frac{1}{15} &= \log\frac{4}{5} - \left(\log\frac{3}{25} + \log\frac{1}{15}\right) \\
&= \log\frac{4}{5} - \log\left(\frac{3}{25}\times\frac{1}{15}\right) \\
&= \log\frac{4}{5} - \log\left(\frac{3}{25\times 15}\right) \\
&= \log\frac{4}{5} - \log\left(\frac{1}{125}\right) \\
&= \log\left(\frac{4}{5}\div\frac{1}{125}\right) \\
&= \log\left(\frac{4}{5}\times\frac{125}{1}\right) \\
&= \log 100 \\
&= \log 10 + \log 10 \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

4. Vini and Dirk complete their mathematics homework and check each other's answers. Compare the two methods shown below and decide if they are correct or incorrect:

Question:

Simplify the following:

$$\log m - \log n - \log p - \log q$$

Vini's answer:

$$\begin{aligned}
\log m - \log n - \log p - \log q &= (\log m - \log n) - \log p - \log q \\
&= \left(\log\frac{m}{n} - \log p\right) - \log q \\
&= \log\left(\frac{m}{n}\times\frac{1}{p}\right) - \log q \\
&= \log\frac{m}{np} - \log q \\
&= \log\frac{m}{np}\times\frac{1}{q} \\
&= \log\frac{m}{npq}
\end{aligned}$$

Dirk's answer:

$$\begin{aligned}
\log m - \log n - \log p - \log q &= \log m - (\log n + \log p + \log q) \\
&= \log m - \log(n\times p\times q) \\
&= \log m - \log(npq) \\
&= \log\frac{m}{npq}
\end{aligned}$$

Solution:

Both methods are correct.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28GR	1b. 28GS	1c. 28GT	1d. 28GV	1e. 28GW	1f. 28GX
2a. 28GY	2b. 28GZ	2c. 28H2	2d. 28H3	2e. 28H4	2f. 28H5
3a. 28H6	3b. 28H7	4. 28H8			

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Simplification of logarithms

Exercise 2 – 16: Simplification of logarithms

Simplify the following without using a calculator:

1. $8^{\frac{2}{3}} + \log_2 32$

   Solution:

$$\begin{aligned} 8^{\frac{2}{3}} + \log_2 32 &= \left(2^3\right)^{\frac{2}{3}} + \log_2 (2^5) \\ &= 2^2 + 5\log_2 2 \\ &= 4 + 5 \\ &= 9 \end{aligned}$$

2. $2\log 3 + \log 2 - \log 5$

   Solution:

$$\begin{aligned} 2\log 3 + \log 2 - \log 5 &= \log 3^2 + \log 2 - \log 5 \\ &= \log \frac{9 \times 2}{5} \\ &= \log \frac{18}{5} \end{aligned}$$

3. $\log_2 8 - \log 1 + \log_4 \frac{1}{4}$

   Solution:

$$\begin{aligned} \log_2 8 - \log 1 + \log_4 \frac{1}{4} &= \log_2 2^3 - 0 + \log_4 4^{(-1)} \\ &= 3\log_2 2 - 1\log_4 4 \\ &= 3(1) - 1(1) \\ &= 2 \end{aligned}$$

4. $\log_8 1 - \log_5 \frac{1}{25} + \log_3 9$

   Solution:

$$\begin{aligned} \log_8 1 - \log_5 \frac{1}{25} + \log_3 9 &= 0 - \log_5 5^{(-2)} + \log_3 3^2 \\ &= -(-2)\log_5 5 + 2\log_3 3 \\ &= 2(1) + 2(1) \\ &= 4 \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28H9 2. 28HB 3. 28HC 4. 28HD

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Solving logarithmic equations

Exercise 2 – 17: Solving logarithmic equations

1. Determine the value of a (correct to 2 decimal places):

   a) $\log_3 a - \log 1{,}2 = 0$

   Solution:

$$\begin{aligned} \log_3 a - \log 1{,}2 &= 0 \\ \log_3 a &= \log 1{,}2 \\ \text{Change to exponential form:} & \\ 3^{\log 1{,}2} &= a \\ \therefore a &= 1{,}09 \end{aligned}$$

   Alternative (longer) method:

$$\begin{aligned} \log_3 a - \log 1{,}2 &= 0 \\ \log_3 a &= \log 1{,}2 \\ \frac{\log a}{\log 3} &= \log 1{,}2 \\ \log a &= \log 3 \times \log 1{,}2 \\ \log a &= 0{,}037\ldots \\ \therefore a &= 1{,}09 \end{aligned}$$

   b) $\log_2 (a - 1) = 1{,}5$

   Solution:

$$\begin{aligned} \log_2 (a-1) &= 1{,}5 \\ \text{Change to exponential form:} & \\ 2^{1{,}5} &= a - 1 \\ 2^{1{,}5} + 1 &= a \\ \therefore a &= 3{,}83 \end{aligned}$$

   Alternative (longer) method:

$$\begin{aligned} \log_2 (a-1) &= 1{,}5 \\ \frac{\log (a-1)}{\log 2} &= 1{,}5 \\ \log (a-1) &= \log 2 \times 1{,}5 \\ \therefore a - 1 &= 2{,}83\ldots \\ \therefore a &= 3{,}83 \end{aligned}$$

   c) $\log_2 a - 1 = 1{,}5$

Solution:

$$\begin{aligned}\log_2 a - 1 &= 1{,}5\\ \log_2 a &= 2{,}5\\ \text{Change to exponential form:}\\ 2^{2{,}5} &= a\\ \therefore a &= 5{,}66\end{aligned}$$

Alternative (longer) method:

$$\begin{aligned}\log_2 a - 1 &= 1{,}5\\ \frac{\log a}{\log 2} &= 2{,}5\\ \log a &= \log 2 \times 2{,}5\\ \therefore a &= 5{,}66\end{aligned}$$

d) $3^a = 2{,}2$

Solution:

$$\begin{aligned}3^a &= 2{,}2\\ \therefore a &= \log_3 2{,}2\\ &= \frac{\log 2{,}2}{\log 3}\\ \therefore a &= 0{,}72\end{aligned}$$

e) $2^{(a+1)} = 0{,}7$

Solution:

$$\begin{aligned}2^{(a+1)} &= 0{,}7\\ \therefore a + 1 &= \log_2 0{,}7\\ \therefore a &= \frac{\log 0{,}7}{\log 2} - 1\\ &= -1{,}51\end{aligned}$$

f) $(1{,}03)^{\frac{a}{2}} = 2{,}65$

Solution:

$$\begin{aligned}(1{,}03)^{\frac{a}{2}} &= 2{,}65\\ \therefore \frac{a}{2} &= \log_{1{,}03} 2{,}65\\ \therefore a &= 2 \times \frac{\log 2{,}65}{\log 1{,}03}\\ &= 65{,}94\end{aligned}$$

g) $(9)^{(1-2a)} = 101$

Solution:

$$\begin{aligned}(9)^{(1-2a)} &= 101\\ \therefore 1 - 2a &= \log_9 101\\ \therefore 1 - \frac{\log 101}{\log 9} &= 2a\\ -1{,}10\ldots &= 2a\\ \therefore -0{,}55 &= a\end{aligned}$$

2. Given $y = 3^x$.

a) Write down the equation of the inverse of $y = 3^x$ in the form $y = \ldots$

Solution:

$y = \log_3 x$

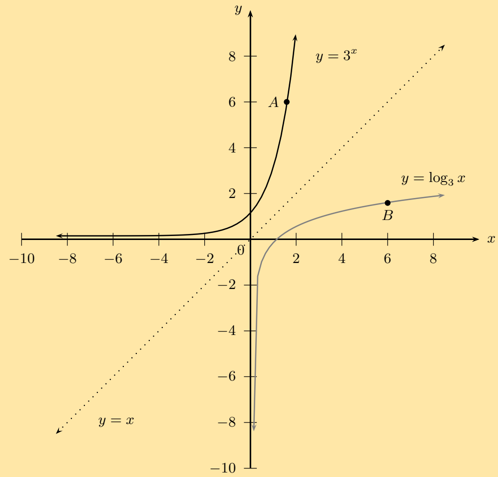
b) If $6 = 3^p$, determine the value of p (correct to one decimal place).

Solution:

$$\begin{aligned} p &= \log_3 6 \\ &= \frac{\log 6}{\log 3} \\ &= 1{,}6 \end{aligned}$$

c) Draw the graph of $y = 3^x$ and its inverse. Plot the points $A(p; 6)$ and $B(6; p)$.

Solution:



Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28HF 1b. 28HG 1c. 28HH 1d. 28HJ 1e. 28HK 1f. 28HM
1g. 28HN 2. 28HP

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Summary

Exercise 2 – 18: Logarithms (ENRICHMENT ONLY)

1. State whether the following are true or false. If false, change the statement so that it is true.

a) $\log t + \log d = \log (t + d)$

Solution:

False: $\log t + \log d = \log (t \times d)$

b) If $p^q = r$, then $q = \log_r p$

Solution:

False: $q = \log_p r$

c) $\log \frac{A}{B} = \log A - \log B$

Solution:
True

d) $\log A - B = \frac{\log A}{\log B}$

Solution:
False: $\log\left(A\right) - B$ cannot be simplified further.

e) $\log_{\frac{1}{2}} x = -\log_2 x$

Solution:
True

f) $\log_k m = \frac{\log_p k}{\log_p m}$

Solution:
False: $\log_k m = \frac{\log_p m}{\log_p k}$

g) $\log_n \sqrt{b} = \frac{1}{2}\log_n b$

Solution:
True

h) $\log_p q = \frac{1}{\log_q p}$

Solution:
True

i) $2\log_2 a + 3\log a = 5\log a$

Solution:
False: bases are different

j) $5\log x + 10\log x = 5\log x^3$

Solution:
True

k) $\frac{\log_n a}{\log_n b} = \log_n \frac{a}{b}$

Solution:
False: cannot be simplified to single logarithm

l) $\log\left(A + B\right) = \log A + \log B$

Solution:
False: do not confuse with $\log\left(AB\right) = \log A + \log B$ or with the distributive law $x(a + b) = ax + ab$.

m) $\log 2a^3 = 3\log 2a$

Solution:
False: $\log 2a^3 = \log 2 + 3\log a$

n) $\frac{\log_n a}{\log_n b} = \log_n\left(a - b\right)$

Solution:
False: do not confuse with $\log_n \left(\frac{a}{b}\right) = \log_n a - \log_n b$. LHS cannot be simplified.

2. Simplify the following without using a calculator:

a) $\log 7 - \log 0{,}7$

Solution:

$$\begin{aligned}\log 7 - \log 0{,}7 &= \log \frac{7}{0{,}7}\\ &= \log 10\\ &= 1\end{aligned}$$

b) $\log 8 \times \log 1$

Solution:

$$\begin{aligned}\log 8 \times \log 1 &= \log 8 \times 0\\ &= 0\end{aligned}$$

c) $\log \frac{1}{3} + \log 300$

Solution:

$$\begin{aligned}\log\frac{1}{3}+\log 300 &= \log\left(\frac{1}{3}\times 300\right)\\ &= \log 100\\ &= \log 10^2\\ &= 2\log 10\\ &= 2\end{aligned}$$

d) $2\log 3+\log 2-\log 6$

Solution:

$$\begin{aligned}2\log 3+\log 2-\log 6 &= \log 3^2+\log 2-\log 6\\ &= \log\frac{9\times 2}{6}\\ &= \log\frac{18}{6}\\ &= \log 3\end{aligned}$$

3. Given $\log 5 = 0{,}7$. Find the value of the following without using a calculator:

a) $\log 50$

Solution:

$$\begin{aligned}\log 50 &= \log 5+\log 10\\ &= 0{,}7+1\\ &= 1{,}7\end{aligned}$$

b) $\log 20$

Solution:

$$\begin{aligned}\log 20 &= \log\frac{100}{5}\\ &= \log 100-\log 5\\ &= 2-0{,}7\\ &= 1{,}3\end{aligned}$$

c) $\log 25$

Solution:

$$\begin{aligned}\log 25 &= \log 5^2\\ &= 2\times\log 5\\ &= 2\times 0{,}7\\ &= 1{,}4\end{aligned}$$

d) $\log_2 5$

Solution:

$$
\begin{aligned}
\log_2 5 &= \frac{\log 5}{\log 2} \\
&= \frac{\log 5}{\log \frac{10}{5}} \\
&= \frac{\log 5}{\log 10 - \log 5} \\
&= \frac{0{,}7}{1 - 0{,}7} \\
&= \frac{0{,}7}{0{,}3} \\
&= \frac{7}{10} \times \frac{10}{3} \\
&= \frac{7}{3}
\end{aligned}
$$

e) $10^{0,7}$

Solution:

If $\log 5 = 0{,}7$, then $10^{0,7} = 5$.

4. Given $A = \log_8 1 - \log_5 \frac{1}{25} + \log_3 9$.

a) Without using a calculator, show that $A = 4$.

Solution:

$$
\begin{aligned}
A &= \log_8 1 - \log_5 \frac{1}{25} + \log_3 9 \\
&= 0 - \log_5 5^{-2} + \log_3 3^2 \\
&= 0 - (-2) \log_5 5 + (2) \log_3 3 \\
&= 2 + 2 \\
&= 4
\end{aligned}
$$

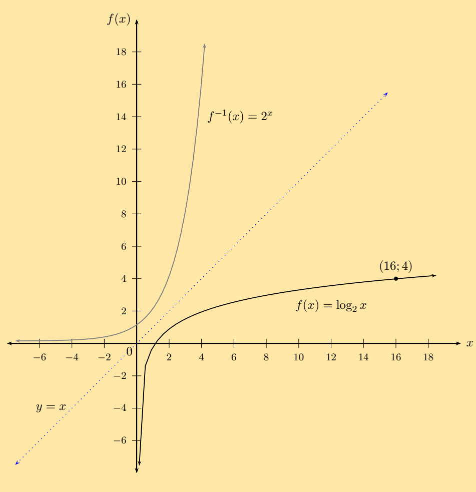
b) Now solve for x if $\log_2 x = A$.

Solution:

$$
\begin{aligned}
\log_2 x &= 4 \\
\therefore x &= 2^4 \\
x &= 16
\end{aligned}
$$

c) Let $f(x) = \log_2 x$. Draw the graph of f and f^{-1}. Indicate the point $(x; A)$ on the graph.

Solution:



5. Solve for x if $\frac{35^x}{7^x} = 15$. Give answer correct to two decimal places.

Solution:

$$\begin{aligned}
\frac{35^x}{7^x} &= 15 \\
\frac{7^x \;.\; 5^x}{7^x} &= 15 \\
5^x &= 15 \\
\therefore x &= \log_5 15 \\
&= \frac{\log 15}{\log 5} \\
&= 1{,}68
\end{aligned}$$

6. Given $f(x) = 5 \times (1{,}5)^x$ and $g(x) = \left(\frac{1}{4}\right)^x$.

   a) For which integer values of x will $f(x) < 295$.

   Solution:

$$\begin{aligned}
5 \times (1{,}5)^x &< 295 \\
\therefore \log (1{,}5)^x &< \log 59 \\
x \log (1{,}5) &< \log 59 \\
x &< \frac{\log 59}{\log (1{,}5)} \qquad \text{note: } \log (1{,}5) > 0 \\
x &< 10{,}0564\ldots
\end{aligned}$$

   Therefore, $x < 10, (x \in \mathbb{Z})$.

   b) For which values of x will $g(x) \geq 2{,}7 \times 10^{-7}$. Give answer to the nearest integer.

   Solution:

$$\begin{aligned}
\left(\frac{1}{4}\right)^x &\geq 2{,}7 \times 10^{-7} \\
\log \left(\frac{1}{4}\right)^x &\geq \log 2{,}7 \times 10^{-7} \\
x \log \left(\frac{1}{4}\right) &\geq \log 2{,}7 \times 10^{-7} \\
x &\leq \frac{\log 2{,}7 \times 10^{-7}}{\log \left(\frac{1}{4}\right)} \qquad \text{note: } \log \left(\frac{1}{4}\right) < 0 \\
&\leq 10{,}9\ldots \\
\therefore x &< 11
\end{aligned}$$

   Important: notice that the inequality sign changed direction when we divided both sides by $\log \left(\frac{1}{4}\right) = -\log 4$, since it has a negative value.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28HQ	1b. 28HR	1c. 28HS	1d. 28HT	1e. 28HV	1f. 28HW
1g. 28HX	1h. 28HY	1i. 28HZ	1j. 28J2	1k. 28J3	1l. 28J4
1m. 28J5	1n. 28J6	2a. 28J7	2b. 28J8	2c. 28J9	2d. 28JB
3a. 28JC	3b. 28JD	3c. 28JF	3d. 28JG	3e. 28JH	4. 28JJ
5. 28JK	6. 28JM				

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