CHAPTER 10

Probability

- This chapter provides good opportunity for experiments and activities in classroom where you can illustrate theoretical probability and number of possible arrangements in practice. Real-life examples have been used extensively in the exercise sections and you may choose to illustrate some of these concepts experimentally in class.
- The terminology and usage of language in this section can be confusing, especially to second-language speakers. Discuss terminology regularly and emphasise the careful reading of questions.
- Union and intersection symbols have been included, but "and" and "or" is the preferred notation in CAPS.
- Make sure to outline the differences between 'and', 'or', 'only' and 'both'. For example, there may be no difference between tea **and** coffee drinkers and tea **or** coffee drinkers in common speech but in probability, the 'and' and 'or' have very specific meanings. Tea **and** coffee drinkers refers to the intersection of tea drinkers, i.e. those who drink both beverages, while tea **or** coffee drinkers refers to the union, i.e. those who drink only tea, those who drink only coffee and those who drink both.
- Some learners may find factorial notation challenging. A lengthy problem-set has been provided to try and iron out some common misconceptions, such as $4!3! \neq 12!$ or $\frac{6!}{4!} \neq \frac{3!}{2!}$, but you may have to go over this more slowly with some learners.
- Note that the formula for the arrangement of n different items in r different places i.e. $\frac{n!}{(n-r)!}$ is NOT included in CAPS and learners should therefore be able to solve these problems logically.
- When applying the fundamental counting principle to probability problems, learners may struggle with knowing when to multiply and when to add probabilities. When a number of different outcomes fit a desired result, the probabilities of each outcome are added. When determining the probability of two or more events occurring, their individual probabilities are multiplied.

10.1 Revision

10.2 Identities

Exercise 10 – 1: The product and addition rules

1. Determine whether the following events are dependent or independent and give a reason for your answer:

   a) Joan has a box of yellow, green and orange sweets. She takes out a yellow sweet and eats it. Then, she chooses another sweet and eats it.

      Solution:

      The two events are dependent because there are fewer sweets to choose from when she picks the second time.

   b) Vuzi throws a die twice.

      Solution:

      The two events are independent because the outcome of the first throw has no effect on the outcome of the second throw.

   c) Celia chooses a card at random from a deck of 52 cards. She is unhappy with her choice, so she places the card back in the deck, shuffles it and chooses a second card.

      Solution:

      The two events are independent because the set of cards in the deck is unchanged each time Celia chooses one randomly.

   d) Thandi has a bag of beads. She randomly chooses a yellow bead, looks at it and then puts it back in the bag. Then she randomly chooses another bead and sees that it is red and puts it back in the bag.

Solution:

The two events are independent because there are the same collection of beads each time Thandi chooses one.

e) Mark has a container with calculators. Some of them work and some are broken. He randomly chooses a calculator and sees that it does not work and throws it away. He then chooses another calculator, sees that it works and keeps it.

Solution:

The two events are dependent because Mark has fewer calculators to choose from when he picks again.

2. Given that $P(A) = 0{,}7$; $P(B) = 0{,}4$ and $P(A \text{ and } B) = 0{,}28$,

a) are events A and B mutually exclusive? Give a reason for your answer.

Solution:

For the events to be mutually exclusive $P(A \text{ and } B)$ must be equal to 0. In this case $P(A \text{ and } B) = 0{,}28$, so the events are not mutually exclusive.

b) are the events A and B independent? Give a reason for your answer.

Solution:

For events to be independent: $P(A) \times P(B) = P(A \text{ and } B)$. $P(A) \times P(B) = 0{,}7 \times 0{,}4 = 0{,}28 = P(A \text{ and } B)$. Therefore the events are independent.

3. In the following examples, are A and B dependent or independent?

a) $P(A) = 0{,}2; P(B) = 0{,}7$ and $P(A \text{ and } B) = 0{,}21$

Solution:

$$P(A) \times P(B) = 0{,}2 \times 0{,}7 = 0{,}14 \neq 0{,}21 = P(A \text{ and } B).$$

Therefore the events are dependent.

b) $P(A) = 0{,}2; P(B) = 0{,}7$ and $P(B \text{ and } A) = 0{,}14$.

Solution:

$$P(A) \times P(B) = 0{,}2 \times 0{,}7 = 0{,}14 = P(B \text{ and } A)$$

Therefore the events are independent.

4. $n(A) = 5; n(B) = 4; n(S) = 20$ and $n(A \text{ or } B) = 8$.

a) Are A and B mutually exclusive?

Solution:

$$P(A) = \frac{5}{20}; P(B) = \frac{4}{20}; P(A \text{ or } B) = \frac{8}{20}$$

For A and B to be mutually exclusive: $P(A) + P(B) = P(A \text{ or } B)$.

$$\frac{5}{20} + \frac{4}{20} = \frac{9}{20} \neq \frac{8}{20}$$

The events are therefore not mutually exclusive.

b) Are A and B independent?

Solution:

For A and B to be independent, $P(A) \times P(B) = P(A \text{ and } B)$

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ \therefore P(A \text{ and } B) &= P(A) + P(B) - P(A \text{ or } B) \\ &= \frac{5}{20} + \frac{4}{20} - \frac{8}{20} \\ &= \frac{1}{20} \\ P(A) \times P(B) &= \frac{1}{4} \times \frac{1}{5} \\ &= \frac{1}{20} = P(A \text{ and } B) \end{aligned}$$

Therefore the events are independent.

5. Simon rolls a die twice. What is the probability of getting:

a) two threes.

Solution:

$$P(\text{two threes}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

b) a prime number then an even number.

Solution:

There are 3 possible prime numbers on a die, namely, 2, 3, and 5, and there are 3 possible even numbers, namely, 2, 4, and 6.

$$\begin{aligned} P(\text{prime number then even number}) &= P(\text{prime number}) \times P(\text{even number}) \\ &= \frac{3}{6} \times \frac{3}{6} = \frac{9}{36} = \frac{1}{4} \end{aligned}$$

c) no threes.

Solution:

If no threes are rolled then, for each of the events, 5 possibilities remain.

$$P(\text{no threes}) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$$

d) only one three.

Solution:

In the sample space there are 36 possible outcomes. There are two ways of getting only one three: either getting a 3 on the first throw and a number other than three on the second throw or a 3 on the second throw and a number other than 3 on the first throw. The outcomes containing only one 3 are: $(3;1)$; $(3;2)$; $(3;4)$; $(3;5)$; $(3;6)$; $(1;3)$; $(2;3)$; $(4;3)$; $(5;3)$; $(6;3)$.

$$P(\text{only one } 3) = \frac{10}{36} = \frac{5}{18}$$

e) at least one three.

Solution:

In the sample space there are 36 possible outcomes. The outcomes containing at least one 3 are: $(3;1)$; $(3;2)$; $(3;3)$; $(3;4)$; $(3;5)$; $(3;6)$; $(1;3)$; $(2;3)$; $(4;3)$; $(5;3)$; $(6;3)$.

$$P(\text{at least one } 3) = \frac{11}{36}$$

6. The Mandalay Secondary soccer team has to win both of their next two matches in order to qualify for the finals. The probability that Mandalay Secondary will win their first soccer match against Ihlumelo High is $\frac{2}{5}$ and the probability of winning their second soccer match against Masiphumelele Secondary is $\frac{3}{7}$. Assume each match is an independent event.

a) What is the probability they will progress to the finals?

Solution:

$$\begin{aligned} P(\text{win and win}) &= \frac{2}{5} \times \frac{3}{7} \\ &= \frac{6}{35} \end{aligned}$$

b) What is the probability they will not win either match?

Solution:

To calculate the probability of not winning a match use:

$$P(\text{not win}) = 1 - P(\text{win})$$

$$\begin{aligned}\text{Therefore } P(\text{not win and not win}) &= \frac{3}{5} \times \frac{4}{7}\\ &= \frac{12}{35}\end{aligned}$$

This solution makes use of the complementary rule which students should be familiar with. We will revise the rule in more detail later.

c) What is the probability they will win only one of their matches?

Solution:

There are two possible outcomes: win-not win or not win-win. Let win $= W$.

$$\begin{aligned}P(\text{(W;not W) or (not W;W)}) &= P(\text{W;not W}) + P(\text{not W;W})\\ &= P(\text{W}) \times P(\text{not W}) + P(\text{not W}) \times P(\text{W})\\ &= \frac{2}{5} \times \frac{4}{7} + \frac{3}{5} \times \frac{3}{7}\\ &= \frac{8}{35} + \frac{9}{35} = \frac{17}{35}\end{aligned}$$

d) You were asked to assume that the matches are independent events but this is unlikely in reality. What are some factors you think may result in the outcome of the matches being dependent?

Solution:

This is an open-ended question designed to get learners to think critically about the dependent or independent nature of real life events. Example answers could include injuries to or suspensions of players during the first match, team morale if they win or lose the first match, etc.

7. A pencil bag contains 2 red pens and 4 green pens. A pen is drawn from the bag and then replaced before a second pen is drawn. Calculate:

a) The probability of drawing a red pen first if a green pen is drawn second.

Solution:

The events are independent so:

$$P(\text{red pen first}) = \frac{2}{6} = \frac{1}{3}$$

b) The probability of drawing a green pen second if the first pen drawn was red.

Solution:

The events are independent so:

$$P(\text{green pen second}) = \frac{4}{6} = \frac{2}{3}$$

c) The probability of drawing a red pen first and a green pen second.

Solution:

$$\begin{aligned}P(\text{first pen red and second pen green}) &= \frac{1}{3} \times \frac{2}{3}\\ &= \frac{2}{9}\end{aligned}$$

8. A lunch box contains 4 sandwiches and 2 apples. Vuyele chooses a food item randomly and eats it. He then chooses another food item randomly and eats that. Determine the following:

a) The probability that the first item is a sandwich.

Solution:

$$P(\text{sandwich first}) = \frac{4}{6} = \frac{2}{3}$$

b) The probability that the first item is a sandwich and the second item is an apple.

Solution:
First item sandwich and second item apple (SA):

$$\frac{4}{6} \times \frac{2}{5} = \frac{8}{30} = \frac{4}{15}$$

c) The probability that the second item is an apple.

Solution:
There are two possible outcomes of getting an apple second:

- first item sandwich and second item apple (SA):

$$= \frac{4}{15} \text{ (from b)}$$

- first item apple and second item apple (AA):

$$\frac{2}{6} \times \frac{1}{5} = \frac{2}{30} = \frac{1}{15}$$

$$\begin{aligned} P(\text{apple second}) &= P(SA) + P(AA) = \frac{4}{15} + \frac{1}{15} \\ &= \frac{1}{3} \end{aligned}$$

d) Are the events in a) and c) dependent? Confirm your answer with a calculation.

Solution:

$$P(\text{sandwich first}) \times P(\text{apple second}) = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9} \neq \frac{4}{15} = P(\text{SA})$$

Therefore the events are dependent.

9. Given that $P(A) = 0{,}5$; $P(B) = 0{,}4$ and $P(A \text{ or } B) = 0{,}7$, determine by calculation whether events A and B are:

a) mutually exclusive

Solution:

$$P(A) + P(B) = 0{,}5 + 0{,}4 = 0{,}9 \neq 0{,}7 = P(A \text{ or } B)$$

Therefore, A and B are not mutually exclusive.

b) independent

Solution:

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ 0{,}7 &= 0{,}5 + 0{,}4 - P(A \text{ and } B) \\ P(A \text{ and } B) &= 0{,}5 + 0{,}4 - 0{,}7 = 0{,}2 \\ P(A) \times P(B) &= 0{,}5 \times 0{,}4 = 0{,}2 = P(A \text{ and } B) \end{aligned}$$

Therefore A and B are independent.

10. A and B are two events in a sample space where $P(A) = 0{,}3$; $P(A \text{ or } B) = 0{,}8$ and $P(B) = k$. Determine the value of k if:

a) A and B are mutually exclusive

Solution:
For A and B to be mutually exclusive: $P(A) + P(B) = P(A \text{ or } B)$

$$\begin{aligned} 0{,}3 + k &= 0{,}8 \\ \therefore k &= 0{,}5 \end{aligned}$$

b) A and B are independent

Solution:
For A and B to be independent:

$$P(A \text{ and } B) = P(A) \times P(B)$$
$$\text{Therefore } P(A \text{ and } B) = 0{,}3k$$

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ 0{,}8 &= 0{,}3 + k - 0{,}3k \\ 0{,}8 &= 0{,}3 + 0{,}7k \\ \therefore 0{,}7k &= 0{,}5 \\ \therefore k &= \frac{5}{7} \end{aligned}$$

11. A and B are two events in sample space S where $n(S) = 36$; $n(A) = 9$; $n(B) = 4$ and $n(\text{not } (A \text{ or } B)) = 24$. Determine:

a) $P(A \text{ or } B)$
Solution:

$$\begin{aligned} P(A \text{ or } B) &= 1 - P(\text{not } (A \text{ or } B)) \\ &= 1 - \frac{24}{36} = \frac{1}{3} \end{aligned}$$

b) $P(A \text{ and } B)$
Solution:

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ \frac{1}{3} &= \frac{9}{36} + \frac{4}{36} - P(A \text{ and } B) \\ \therefore P(A \text{ and } B) &= \frac{9}{36} + \frac{4}{36} - \frac{1}{3} \\ &= \frac{1}{36} \end{aligned}$$

c) whether events A and B independent. Justify your answer with a calculation.
Solution:
For independent events $P(A) \times P(B) = P(A \text{ and } B)$

$$P(A) \times P(B) = \frac{1}{4} \times \frac{1}{9} = \frac{1}{36} = P(A \text{ and } B)$$

Therefore A and B are independent.

12. The probability that a Mathematics teacher is absent from school on a certain day is $0{,}2$. The probability that the Science teacher will be absent that same day is $0{,}3$.

a) Do you think these two events are independent? Give a reason for your answer.
Solution:
Learner dependent. For example: No, there could a bug or illness spreading through the school, therefore the absence of both teachers may be dependent.

b) Assuming the events are independent, what is the probability that the Mathematics teacher or the Science teacher is absent?
Solution:
Let the probability that the Mathematics teacher is absent $= P(M)$ and the probability that the Science teacher is absent $= P(S)$.

$$P(M \text{ or } S) = P(M) + P(S) - P(M \text{ and } S)$$

Assuming the events are independent:

$$\begin{aligned} P(M \text{ and } S) &= 0{,}2 \times 0{,}3 = 0{,}06 \\ \text{Therefore } P(M \text{ or } S) &= 0{,}2 + 0{,}3 - 0{,}06 \\ &= 0{,}44 \end{aligned}$$

c) What is the probability that neither the Mathematics teacher nor the Science teacher is absent?

Solution:

$$P(\text{not } (M \text{ or } S)) = 1 - 0{,}44 = 0{,}56$$

13. Langa Cricket Club plays two cricket matches against different clubs. The probability of winning the first match is $\frac{3}{5}$ and the probability of winning the second match is $\frac{4}{9}$. Assuming the results of the matches are independent, calculate the probability that Langa Cricket Club will:

a) win both matches.

Solution:

Let $P(M)$ = the probability of winning the first match and $P(N)$ = the probability of winning the second match.

$$\begin{aligned} P(M \text{ and } N) &= \frac{3}{5} \times \frac{4}{9} \\ &= \frac{12}{45} \\ &= \frac{4}{15} \end{aligned}$$

b) not win the first match.

Solution:

$$\begin{aligned} P(\text{not } M) &= 1 - \frac{3}{5} \\ &= \frac{2}{5} \end{aligned}$$

c) win one or both of the two matches.

Solution:

$$\begin{aligned} P(M \text{ or } N) &= P(M) + P(N) - P(M \text{ and } N) \\ &= \frac{3}{5} + \frac{4}{9} - \frac{4}{15} \\ &= \frac{7}{9} \end{aligned}$$

d) win neither match.

Solution:

$$\begin{aligned} P(\text{not } M \text{ and not } N) &= P(\text{not } M) \times P(\text{not } N) \\ &= \left(1 - \frac{3}{5}\right) \times \left(1 - \frac{4}{9}\right) \\ &= \frac{2}{5} \times \frac{5}{9} \\ &= \frac{2}{9} \end{aligned}$$

e) not win the first match and win the second match.

Solution:

$$\begin{aligned} P(\text{not } M \text{ and } N) &= P(\text{not } M) \times P(N) \\ &= \frac{2}{5} \times \frac{4}{9} \\ &= \frac{8}{45} \end{aligned}$$

14. Two teams are working on the final problem at a Mathematics Olympiad. They have 10 minutes remaining to finish the problem. The probability that team A will finish the problem in time is 40% and the probability that team B will finish the problem in time is 25%. Calculate the probability that both teams will finish before they run out of time.

Solution:

Let the probability that team A will finish $= P(A)$ and the probability team B will finish $= P(B)$. The teams are working separately, therefore the two events are independent.

$$\begin{aligned} P(A \text{ and } B) &= P(A) \times P(B) \\ &= 0.4 \times 0.25 \\ &= 0.1 \text{ or } 10\% \end{aligned}$$

15. Thabo and Julia were arguing about whether people prefer tea or coffee. Thabo suggested that they do a survey to settle the dispute. In total, they surveyed 24 people and found that 8 of them preferred to drink coffee and 12 of them preferred to drink tea. The number of people who drink tea, coffee or both is 16. Determine:

a) the probability that a person drinks tea, coffee or both.

Solution:

Let $n(C)$ be the number of people who drink coffee and $n(T)$ be the number of people who drink tea.

$$\begin{aligned} P(C \text{ or } T) &= \frac{n(C \text{ or } T)}{n(S)} \\ &= \frac{16}{24} \\ &= \frac{2}{3} \end{aligned}$$

b) the probability that a person drinks neither tea nor coffee.

Solution:

$$\begin{aligned} P(\text{not } (C \text{ or } T)) &= 1 - P(C \text{ or } T) \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

c) the probability that a person drinks coffee and tea.

Solution:

$$\begin{aligned} P(C \text{ and } T) &= P(C) + P(T) - P(C \text{ or } T) \\ &= \frac{n(C)}{n(S)} + \frac{n(T)}{n(S)} - \frac{n(C \text{ or } T)}{n(S)} \\ &= \frac{8}{24} + \frac{12}{24} - \frac{16}{24} \\ &= \frac{1}{6} \end{aligned}$$

d) the probability that a person does not drink coffee.

Solution:

$$\begin{aligned} P(\text{not } C) &= 1 - P(C) \\ &= 1 - \frac{8}{24} \\ &= \frac{2}{3} \end{aligned}$$

e) whether the event that a person drinks coffee and the event that a person drinks tea are independent.

Solution:

$$\begin{aligned} P(C) \times P(T) &= \frac{1}{3} \times \frac{1}{2} \\ &= \frac{1}{6} = P(C \text{ and } T) \end{aligned}$$

Therefore the events are independent.

Check answers online with the exercise code below or click on 'show me the answer'.

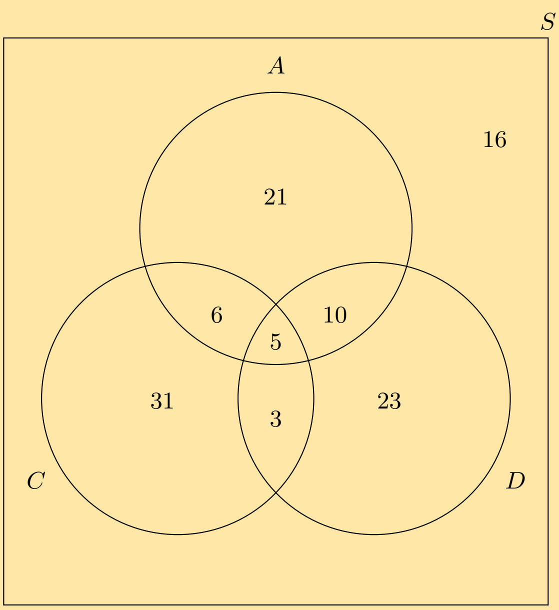
1. 29FV 2. 29FW 3. 29FX 4. 29FY 5. 29FZ 6. 29G2
7. 29G3 8. 29G4 9. 29G5 10. 29G6 11. 29G7 12. 29G8
13. 29G9 14. 29GB 15. 29GC

www.everythingmaths.co.za m.everythingmaths.co.za

10.3 Tools and Techniques

Exercise 10 – 2: Venn and tree diagrams

1. A survey was done on a group of learners to determine which type of TV shows they enjoy: action, comedy or drama. Let A = action, C = comedy and D = drama. The results of the survey are shown in the Venn diagram below.



Study the Venn diagram and determine the following:

a) the total number of learners surveyed

Solution:

115

b) the number of learners who do not enjoy any of the mentioned types of TV shows

Solution:

16

c) $P(\text{not } A)$

Solution:

$\frac{73}{115}$

d) $P(A \text{ or } D)$

Solution:

$\frac{68}{115}$

e) $P(A \text{ and } C \text{ and } D)$

Solution:

$\frac{5}{115} = \frac{1}{23}$

f) $P(\text{not } (A \text{ and } D))$

Solution:

$\frac{100}{115} = \frac{20}{23}$

g) $P(A \text{ or not } C)$

Solution:

$\frac{81}{115}$

h) $P(\text{not } (A \text{ or } C))$

Solution:

$\frac{39}{115}$

i) the probability of a learner enjoying at least two types of TV shows

Solution:

$\frac{24}{115}$

j) Describe, in words, the meaning of each of the questions c) to h) in the context of this problem.

Solution:

$P(\text{not } A)$: the probability that learners do not enjoy action TV shows

$P(A \text{ or } D)$: the probability that learners enjoy action or drama TV shows

$P(A \text{ and } D \text{ and } C)$: the probability that learners enjoy action, drama and comedy TV shows

$P(\text{not } (A \text{ and } D))$: the probability that learners do not enjoy action and drama TV shows

$P(A \text{ or not } C)$: the probability that learners enjoy action TV shows or do not enjoy comedy TV shows

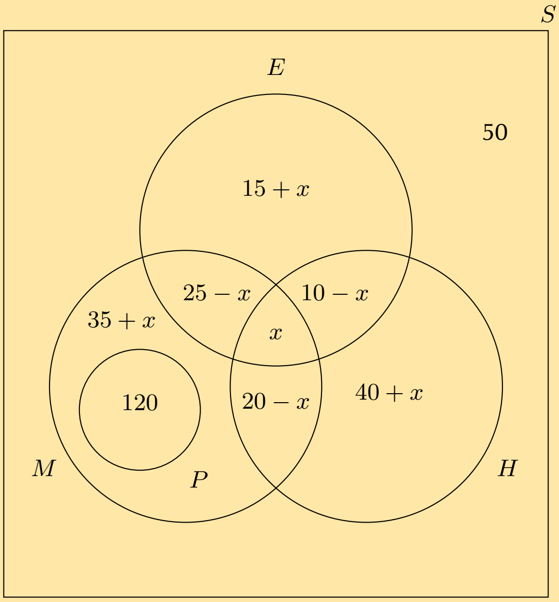
$P(\text{not } (A\text{or } C))$: the probability that learners do not enjoy action or comedy TV shows

2. At Thandokulu Secondary School, there are are 320 learners in Grade 12, 270 of whom take one or more of Mathematics, History and Economics. The subject choice is such that everybody who takes Physical Sciences must also take Mathematics and nobody who takes Physical Sciences can take History or Economics. The following is known about the number of learners who take these subjects:

- 70 take History
- 50 take Economics
- 120 take Physical Sciences
- 200 take Mathematics
- 20 take Mathematics and History
- 10 take History and Economics
- 25 take Mathematics and Economics
- x learners take Mathematics and History and Economics

a) Represent the information above in a Venn diagram. Let Mathematics be M, History be H, Physical Sciences be P and Economics be E.

Solution:



b) Determine the number of learners, x, who take Mathematics, History and Economics.

Solution:

$$
\begin{aligned}
120 + (35 + x) + (25 - x) + (20 - x) + x + (40 + x) + (10 - x) + (15 + x) &= 270 \\
265 + x &= 270 \\
x &= 5
\end{aligned}
$$

Therefore 5 learners take Mathematics, History and Economics.

c) Determine $P(\text{not } (M \text{ or } H \text{ or } E))$ and state in words what your answer means.

Solution:

$$P(\text{not } (M \text{ or } H \text{ or } E)) = \frac{50}{320} = \frac{5}{32}$$

This is the probability that a learner does not take Mathematics, History or Economics.

d) Determine the probability that a learner takes at least two of these subjects.

Solution:

This question requires us to find the sum of the probabilities of all the learners who take at least two subjects. This includes the intersection of each of the subjects.

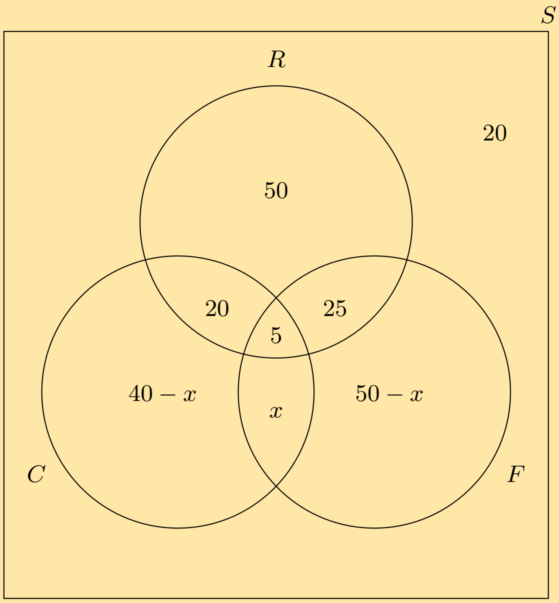
$$
\begin{aligned}
P(\text{at least two subjects}) &= \frac{120 + 20 + 5 + 5 + 15}{320} \\
&= \frac{33}{64}
\end{aligned}
$$

3. A group of 200 people were asked about the kind of sports they watch on television. The information collected is given below:

- 180 watch rugby, cricket or soccer
- 5 watch rugby, cricket and soccer
- 25 watch rugby and cricket
- 30 watch rugby and soccer
- 100 watch rugby
- 65 watch cricket
- 80 watch soccer
- x watch cricket and soccer but not rugby

a) Represent all the above information in a Venn diagram. Let rugby watchers $= R$, cricket watchers $= C$ and soccer watchers $= F$.

Solution:



b) Find the value of x.

Solution:

$$
\begin{aligned}
50 + 25 + 5 + 20 + (40 - x) + (50 - x) + x &= 180 \\
190 - x &= 180 \\
\text{Therefore } x &= 10
\end{aligned}
$$

c) Determine $P(\text{not } (R \text{ or } F \text{ or } C))$

Solution:

$$P(\text{not } (R \text{ or } F \text{ or } C)) = \frac{20}{200} = \frac{1}{10}$$

d) Determine $P(R \text{ or } F \text{ or not } C)$

Solution:

$$P(R \text{ or } F \text{ or not } C) = \frac{170}{200} = \frac{17}{20}$$

e) Are watching cricket and watching rugby independent events? Confirm your answer using a calculation.

Solution:

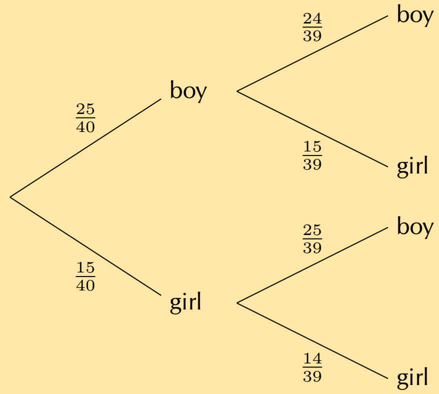
$$
\begin{aligned}
P(R) &= \frac{100}{200} = \frac{1}{2} \\
P(C) &= \frac{65}{200} = \frac{13}{40} \\
P(R \text{ and } C) &= \frac{25}{200} = \frac{1}{8} \\
P(R) \times P(C) &= \frac{1}{2} \times \frac{13}{40} = \frac{13}{80} \neq \frac{1}{8} = P(R \text{ and } C)
\end{aligned}
$$

Therefore watching rugby and watching cricket are dependent events.

4. There are 25 boys and 15 girls in the English class. Each lesson, two learners are randomly chosen to do an oral.

   a) Represent the composition of the English class in a tree diagram. Include all possible outcomes and probabilities.

Solution:



b) Calculate the probability that a boy and a girl are chosen to do an oral in any particular lesson.

Solution:

$$\begin{aligned}\left(\frac{25}{40}\times\frac{15}{39}\right)+\left(\frac{15}{40}\times\frac{25}{39}\right) &= \frac{25}{104}+\frac{25}{104}\\ &= \frac{25}{52}\end{aligned}$$

c) Calculate the probability that at least one of the learners chosen to do an oral in any particular lesson is male.

Solution:

Note: This question can be answered by subtracting the outcome not containing a boy (girl; girl) from 1 (shown below) or by adding the three outcomes which include a boy. Either method is correct.

$$\begin{aligned}1-\left(\frac{15}{40}\times\frac{14}{39}\right) &= 1-\frac{7}{52}\\ &= \frac{45}{52}\end{aligned}$$

d) Are the events picking a boy first and picking a girl second independent or dependent? Justify your answer with a calculation.

Solution:

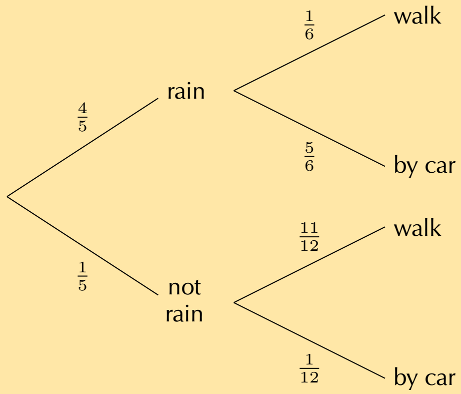
$$\begin{aligned}P(\text{boy first}) &= \frac{25}{40}=\frac{5}{8}\\ P(\text{girl second}) &= \left(\frac{15}{40}\times\frac{14}{39}\right)+\left(\frac{25}{40}\times\frac{15}{39}\right)\\ &= \frac{7}{52}+\frac{25}{104}\\ &= \frac{3}{8}\\ P(\text{boy first})\times P(\text{girl second}) &= \frac{5}{8}\times\frac{3}{8}=\frac{15}{64}\\ P(\text{boy first and girl second}) &= \frac{25}{40}\times\frac{15}{39}\\ &= \frac{25}{104}\neq\frac{15}{64}=P(\text{boy first})\times P(\text{girl second})\end{aligned}$$

Therefore picking a boy first and picking a girl second are dependent events.

5. During July in Cape Town, the probability that it will rain on a randomly chosen day is $\frac{4}{5}$. Gladys either walks to school or gets a ride with her parents in their car. If it rains, the probability that Gladys' parents will take her to school by car is $\frac{5}{6}$. If it does not rain, the probability that Gladys' parents will take her to school by car is $\frac{1}{12}$.

a) Represent the above information in a tree diagram. On your diagram show all the possible outcomes and respective probabilities.

Solution:



b) What is the probability that it is a rainy day and Gladys walks to school?

Solution:

$$\begin{aligned} P(\text{rain and walk}) &= \frac{4}{5} \times \frac{1}{6} \\ &= \frac{2}{15} \end{aligned}$$

c) What is the probability that Gladys' parents take her to school by car?

Solution:

$$\begin{aligned} P(\text{by car}) &= P(\text{rain and car}) + P(\text{no rain and car}) \\ &= \left(\frac{4}{5} \times \frac{5}{6}\right) + \left(\frac{1}{5} \times \frac{1}{12}\right) \\ &= \frac{2}{3} + \frac{1}{60} \\ &= \frac{41}{60} \end{aligned}$$

6. There are two types of property burglaries: burglary of private residences and burglary of business premises. In Metropolis, burglary of a private residence is four times as likely as that of a business premises. The following statistics for each type of burglary were obtained from the Metropolis Police Department:

Burglary of private residences

Following a burglary:

- 25% of criminals are arrested within 48 hours.
- 15% of criminals are arrested after 48 hours.
- 60% of criminals are never arrested for that particular burglary.

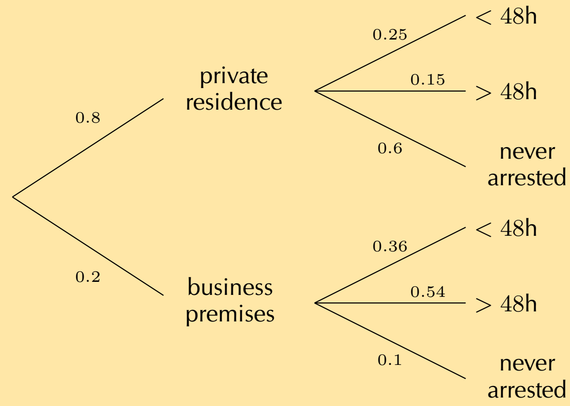
Burglary of business premises

Following a burglary:

- 36% of criminals are arrested within 48 hours.
- 54% of criminals are arrested after 48 hours.
- 10% of criminals are never arrested for that particular burglary.

a) Represent the information above in a tree diagram, showing all outcomes and respective probabilities.

Solution:



b) Calculate the probability that a private home is burgled and nobody is arrested.

Solution:

$$\begin{aligned} P(\text{private home and never arrested}) &= 0{,}8 \times 0{,}6 \\ &= 0{,}48 \end{aligned}$$

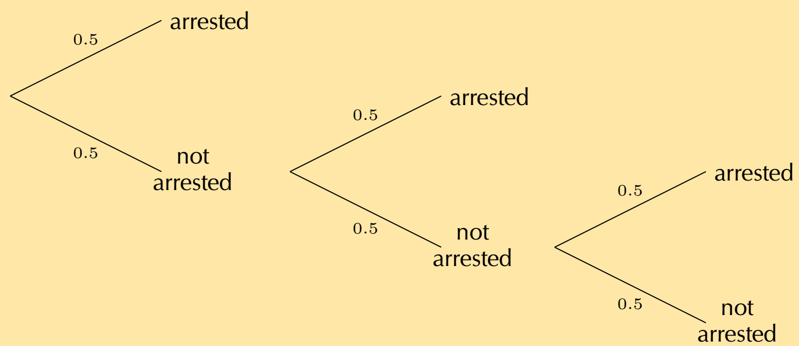
c) Calculate the probability that burglars of private homes and business premises are arrested.

Solution:

$$\begin{aligned} P(\text{arrested}) &= (0{,}8 \times 0{,}25) + (0{,}8 \times 0{,}15) + (0{,}2 \times 0{,}36) + (0{,}2 \times 0{,}54) \\ &= 0{,}5 \end{aligned}$$

d) Use your answer in the previous question to construct a tree diagram to calculate the probability that a burglar is arrested after at most three burglaries.

Solution:



$$\begin{aligned} P(\text{arrested after 3 burglaries}) &= 0{,}5 + (0{,}5 \times 0{,}5) + (0{,}5 \times 0{,}5 \times 0{,}5) \\ &= 0{,}875 \end{aligned}$$

This answer could also be reached by subtracting the probability of not being arrested after three burglaries from 1:

$$1 - (0{,}5 \times 0{,}5 \times 0{,}5) = 1 - 0{,}5^3 = 1 - 0{,}125 = 0{,}875$$

We will use this principle to answer the next question.

e) Determine after how many burglaries a burglar has at least a

i. 90% chance of being arrested.
ii. 99% chance of being arrested.

Solution:

Let the number of burglaries $= n$

i.

$$\begin{aligned} 0{,}90 &= 1 - P(\text{not arrested})^n \\ &= 1 - 0{,}5^n \\ \text{Therefore } 0{,}1 &= 0{,}5^n \\ \text{Therefore } n &= \log_{0{,}5} 0{,}1 \\ &= 3{,}32 \end{aligned}$$

After 4 burglaries, there will be at least a 90% chance of being arrested.

ii.

$$\begin{aligned} 0{,}99 &= 1 - P(\text{not arrested})^n \\ &= 1 - 0{,}5^n \\ \text{Therefore } 0{,}01 &= 0{,}5^n \\ \text{Therefore } n &= \log_{0{,}5} 0{,}01 \\ &= 6{,}64 \end{aligned}$$

After 7 burglaries, there will be at least a 99% chance of being arrested.

Check answers online with the exercise code below or click on 'show me the answer'.
1. 29GF 2. 29GG 3. 29GH 4. 29GJ 5. 29GK 6. 29GM

www.everythingmaths.co.za m.everythingmaths.co.za

Exercise 10 – 3: Contingency tables

1. A number of drivers were asked about the number of motor vehicle accidents they were involved in over the last 10 years. Part of the data collected is shown in the table below.

	≤ 2 accidents	> 2 accidents	Total
Female	210	90	
Male			
Total	350	150	500

a) What are the variables investigated here and what is the purpose of the research?

Solution:

The variables are gender and number of accidents over a period of 10 years. The purpose of the research is to determine if gender is related to the number of accidents a driver is involved in.

b) Complete the table.

Solution:

	≤ 2 accidents	> 2 accidents	Total
Female	210	90	300
Male	140	60	200
Total	350	150	500

c) Determine whether gender and number of accidents are independent suing a calculation.

Solution:

$$P(\text{female}) = \frac{300}{500} = 0{,}6$$
$$P(\text{male}) = \frac{200}{500} = 0{,}4$$
$$P(\leq 2 \text{ accidents}) = \frac{350}{500} = 0{,}7$$
$$P(> 2 \text{ accidents}) = \frac{150}{500} = 0{,}3$$
$$P(\text{female and } \leq 2 \text{ accidents}) = \frac{210}{500} = 0{,}42$$
$$P(\text{female and } > 2 \text{ accidents}) = \frac{90}{500} = 0{,}18$$
$$P(\text{male and } \leq 2 \text{ accidents}) = \frac{140}{500} = 0{,}28$$
$$P(\text{male and } > 2 \text{ accidents}) = \frac{60}{500} = 0{,}12$$
$$P(\text{female}) \times P(\leq 2 \text{ accidents}) = 0{,}42 = P(\text{female and } \leq 2 \text{ accidents})$$
$$P(\text{female}) \times P(> 2 \text{ accidents}) = 0{,}18 = P(\text{female and } > 2 \text{ accidents})$$
$$P(\text{male}) \times P(\leq 2 \text{ accidents}) = 0{,}28 = P(\text{male and } \leq 2 \text{ accidents})$$
$$P(\text{male}) \times P(> 2 \text{ accidents}) = 0{,}12 = P(\text{male and } > 2 \text{ accidents})$$

It can be seen that in all cases $P(A) \times P(B) = P(A \text{ and } B)$, therefore number of motor vehicle accidents is independent of the gender of the driver.

2. Researchers conducted a study to test how effective a certain inoculation is at preventing malaria. Part of their data is shown below:

	Malaria	**No malaria**	**Total**
Male	a	b	216
Female	c	d	648
Total	108	756	864

a) Calculate the probability that a randomly selected study participant will be female.

Solution:

$P(\text{female}) = \frac{648}{864} = \frac{3}{4}$

b) Calculate the probability that a randomly selected study participant will have malaria.

Solution:

$P(\text{malaria}) = \frac{108}{864} = \frac{1}{8}$

c) If being female and having malaria are independent events, calculate the value c.

Solution:

$$P(\text{female and malaria}) = \frac{3}{4} \times \frac{1}{8} = \frac{3}{32}$$
$$\therefore c = \frac{3}{32} \times 864 = 81$$

d) Using the value of c, fill in the missing values on the table.

Solution:

	Malaria	**No malaria**	**Total**
Male	27	189	216
Female	81	567	648
Total	108	756	864

3. The reaction time of 400 drivers during an emergency stop was tested. Within the study cohort (the group of people being studied), the probability that a driver chosen at random was 40 years old or younger is 0,3 and the probability of a reaction time less than 1,5 seconds is 0,7.

a) Calculate the number of drivers who are 40 years old or younger.

Solution:

$n(\text{forty years and younger}) = 0{,}3 \times 400 = 120$

b) Calculate the number of drivers who have a reaction time of less than $1{,}5$ seconds.

Solution:

$n(\text{reaction time } < 1{,}5 \text{ s}) = 0{,}7 \times 400 = 280$

c) If age and reaction time are independent events, calculate the number of drivers 40 years old and younger with a reaction time of less than $1{,}5$ seconds.

Solution:

$$P(40 \text{ or younger and reaction time } < 1{,}5 \text{ secs}) = 0{,}3 \times 0{,}7 = 0{,}21$$
$$\therefore n(40 \text{ or younger and reaction time } < 1{,}5 \text{ secs}) = 0{,}21 \times 400 = 84$$

d) Complete the table below.

	Reaction time $< 1{,}5$ s	**Reaction time $> 1{,}5$ s**	**Total**
≤ 40 years			
> 40 years			
Total			400

Solution:

	Reaction time $< 1{,}5$ s	**Reaction time $> 1{,}5$ s**	**Total**
≤ 40 years	84	36	120
> 40 years	196	84	280
Total	280	120	400

4. A new treatment for influenza (the flu) was tested on a number of patients to determine if it was better than a placebo (a pill with no therapeutic value). The table below shows the results three days after treatment:

	Flu	**No flu**	**Total**
Placebo	228	60	
Treatment			
Total	240	312	

a) Complete the table.

Solution:

	Flu	**No flu**	**Total**
Placebo	228	60	288
Treatment	12	252	264
Total	240	312	552

b) Calculate the probability of a patient receiving the treatment.

Solution:

$$P(\text{treatment}) = \frac{n(\text{treatment})}{n(\text{total patients})} = \frac{312}{552} = \frac{13}{23}$$

c) Calculate the probability of a patient having no flu after three days.

Solution:

$$P(\text{no flu}) = \frac{n(\text{no flu})}{n(\text{total patients})} = \frac{264}{552} = \frac{11}{23}$$

d) Calculate the probability of a patient receiving the treatment and having no flu after three days.

Solution:

$$\begin{aligned} P(\text{no flu and treatment}) &= \frac{n(\text{no flu and treatment})}{n(\text{total patients})} \\ &= \frac{252}{552} = \frac{21}{46} \end{aligned}$$

e) Using a calculation, determine whether a patient receiving the treatment and having no flu after three days are dependent or independent events.

Solution:

$$\begin{aligned} P(\text{treatment}) \times P(\text{no flu}) &= \frac{11}{23} \times \frac{13}{23} = \frac{143}{529} = 0{,}270 \\ P(\text{treatment and no flu} &= \frac{21}{46} = 0{,}457 \end{aligned}$$

Therefore receiving treatment and having no flu after three days are dependent events.

f) Calculate the probability that a patient receiving treatment will have no flu after three days.

Solution:

$$\begin{aligned} P(\text{no flu if treated}) &= \frac{n(\text{no flu and treatment})}{n(\text{total treated})} \\ &= \frac{252}{264} = \frac{21}{22} \end{aligned}$$

g) Calculate the probability that a patient receiving a placebo will have no flu after three days.

Solution:

$$\begin{aligned} P(\text{no flu if given placebo}) &= \frac{n(\text{no flu and placebo})}{n(\text{total placebo})} \\ &= \frac{60}{288} = \frac{5}{24} \end{aligned}$$

h) Comparing you answers in f) and g), would you recommend the use of the new treatment for patients suffering from influenza?

Solution:

The probability of having no influenza after three days is much higher when on the new treatment so its use is recommended.

i) A hospital is trying to decide whether to purchase the new treatment. The new treatment is much more expensive than the old treatment. According to the hospital records, of the 72 024 flu patients that have been treated with the old treatment, only 3200 still had the flu three days after treatment.

- Construct a two-way contingency table comparing the old treatment data with the new treatment data.
- Using the data from your table, advise the hospital whether to purchase the new treatment or not.

Solution:

	Flu	**No flu**	**Total**
Old treatment	3200	68 824	72 024
New treatment	12	252	264
Total	3212	69 076	72 288

$$\begin{aligned} P(\text{no flu if old treatment}) &= \frac{68\,824}{72\,024} \\ &= \frac{8603}{9003} = 0{,}956 \\ P(\text{no flu if new treatment}) &= \frac{252}{264} == 0{,}955 \end{aligned}$$

The probability of not having flu after three days if given the new treatment is approximately the same if given the old treatment, therefore the hospital should not purchase the new, more expensive treatment.

5. Human immunodeficiency virus (HIV) affects 10% of the South African population.

   a) If a test for HIV has a $99{,}9\%$ accuracy rate (i.e. $99{,}9\%$ of the time the test is correct, $0{,}1\%$ of the time, the test returns a false result), draw a two-way contingency table showing the expected results if $10\ 000$ of the general population are tested.

   Solution:

   If $10\ 000$ people are tested and the prevalence rate is 10%:

$$\begin{aligned} 10\ 000 \times 0{,}1 &= 1000 \text{ people are expected to be sick} \\ \text{Therefore } 10\ 000 - 1000 &= 9000 \text{ people are expected to be healthy} \end{aligned}$$

	Sick	Healthy	Total
Positive			
Negative			
Total	1000	9000	10 000

   If the test is $99{,}9\%$ accurate:

$$\begin{aligned} 1000 \times 0{,}999 &= 999 \text{ sick people are expected to test positive} \\ \text{Therefore } 1000 - 999 &= 1 \text{ sick person is expected to test negative} \\ \text{And } 9000 \times 0{,}999 &= 8991 \text{ healthy people are expected to test negative} \\ \text{Therefore } 9000 - 8991 &= 9 \text{ healthy people are expected to test positive} \end{aligned}$$

	Sick	Healthy	Total
Positive	999	9	1008
Negative	1	8991	8992
Total	1000	9000	10 000

   b) Calculate the probability that a person who tests positive for HIV does not have the disease, correct to two decimal places.

   Solution:

$$\begin{aligned} P(\text{healthy if tested positive}) &= \frac{n(\text{healthy and tested positive})}{n(\text{tested positive})} \\ &= \frac{9}{1008} \\ &= 0{,}01 \end{aligned}$$

   It is worth noting that this probability is bigger than the one suggested by the '$99{,}9\%$ accuracy' of the test.

   c) In practice, a person who tests positive for HIV is always tested a second time. Calculate the probability that an HIV-negative person will test positive after two tests, correct to four decimal places.

   Solution:

$$\begin{aligned} P(\text{healthy if tested positive twice}) &= \frac{9}{1008} \times \frac{9}{1008} \\ &= \frac{81}{1\ 016\ 064} \\ &= 0{,}0001 \end{aligned}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29GN 2. 29GP 3. 29GQ 4. 29GR 5. 29GS

www.everythingmaths.co.za m.everythingmaths.co.za

Exercise 10 – 4: Number of possible outcomes if repetition is allowed

1. Tarryn has five different skirts, four different tops and three pairs of shoes. Assuming that all the colours complement each other, how many different outfits can she put together?

   Solution:

   $$5 \times 4 \times 3 = 60 \text{ different outfits}$$

2. In a multiple-choice question paper of 20 questions the answers can be A, B, C or D. How many different ways are there of answering the question paper?

   Solution:

   $$4^{20} = 1{,}0995 \times 10^{12} \text{ different ways of answering the exam paper}$$

3. A debit card requires a five digit personal identification number (PIN) consisting of digits from 0 to 9. The digits may be repeated. How many possible PINs are there?

   Solution:

   $$10^5 = 100\ 000 \text{ possible PINs}$$

4. The province of Gauteng ran out of unique number plates in 2010. Prior to 2010, the number plates were formulated using the style LLLDDDGP, where L is any letter of the alphabet excluding vowels and Q, and D is a digit between 0 and 9. The new style the Gauteng government introduced is LLDDLLGP. How many more possible number plates are there using the new style when compared to the old style?

   Solution:

   $$\text{Old style: } 20^3 \times 10^3 = 8\ 000\ 000 \text{ possible arrangements}$$
   $$\text{New style: } 20^4 \times 10^2 = 16\ 000\ 000 \text{ possible arrangements}$$
   $$16\ 000\ 000 - 8\ 000\ 000 = 8\ 000\ 000$$

   Therefore there are $8\ 000\ 000$ more possible number plates using the new style.

5. A gift basket is made up from one CD, one book, one box of sweets, one packet of nuts and one bottle of fruit juice. The person who makes up the gift basket can choose from five different CDs, eight different books, three different boxes of sweets, four kinds of nuts and six flavours of fruit juice. How many different gift baskets can be produced?

   Solution:

   $$5 \times 8 \times 3 \times 4 \times 6 = 2880 \text{ possible gift baskets}$$

6. The code for a safe is of the form XXXXYYY where X is any number from 0 to 9 and Y represents the letters of the alphabet. How many codes are possible for each of the following cases:

   a) the digits and letters of the alphabet can be repeated.
      Solution:

      $$10^4 \times 26^3 = 175\ 760\ 000 \text{ possible codes}$$

   b) the digits and letters of the alphabet can be repeated, but the code may not contain a zero or any of the vowels in the alphabet.
      Solution:
      We exclude the digit 0 and the vowels (A; E; I; O; U), leaving 9 other digits and 21 letters to choose from.

      $$9^4 \times 21^3 = 60\ 761\ 421 \text{ possible codes}$$

c) the digits and letters of the alphabet can be repeated, but the digits may only be prime numbers and the letters X, Y and Z are excluded from the code.

Solution:

The prime digits are 2, 3, 5 and 7. This gives us 4 possible digits. If we exclude the letters X, Y and Z, we are left with 23 letters to choose from.

$$4^4 \times 23^3 = 3\,114\,752 \text{ possible codes}$$

7. A restaurant offers four choices of starter, eight choices for the main meal and six choices for dessert. A customer can choose to eat just one course, two different courses or all three courses. Assuming that all courses are available, how many different meal options does the restaurant offer?

Solution:

- A person who eats only a starter has 4 choices
- A person who eats only a main meal has 8 choices
- A person who eats only a dessert has 6 choices
- A person who eats a starter and a main course has $4 \times 8 = 32$ choices
- A person who eats a starter and a dessert has $4 \times 6 = 24$ choices
- A person who eats a main meal and a dessert has $8 \times 6 = 48$ choices
- A person who eats all three courses has $4 \times 8 \times 6 = 192$ choices.

Therefore, there are $4 + 8 + 6 + 32 + 24 + 48 + 192 = 314$ different meal options

Check answers online with the exercise code below or click on 'show me the answer'.
1. 29GV 2. 29GW 3. 29GX 4. 29GY 5. 29GZ 6. 29H2
7. 29H3

www.everythingmaths.co.za m.everythingmaths.co.za

10.5 Factorial notation

Exercise 10 – 5: Factorial notation

1. Work out the following without using a calculator:

a) $3!$

Solution:

$$3 \times 2 \times 1 = 6$$

b) $6!$

Solution:

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

c) $2!3!$

Solution:

$$2 \times 1 \times 3 \times 2 \times 1 = 12$$

d) $8!$

Solution:

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40\,320$$

e) $\frac{6!}{3!}$

Solution:

$$\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 120$$

f) $6! + 4! - 3!$

Solution:

$$(6 \times 5 \times 4 \times 3 \times 2 \times 1) + (4 \times 3 \times 2 \times 1) - (3 \times 2 \times 1) = 720 + 24 - 6 = 738$$

g) $\frac{6! - 2!}{2!}$

Solution:

$$\frac{(6 \times 5 \times 4 \times 3 \times 2 \times 1) - (2 \times 1)}{2 \times 1} = \frac{720 - 2}{2} = 359$$

h) $\frac{2! + 3!}{5!}$

Solution:

$$\frac{(2 \times 1) + (3 \times 2 \times 1)}{5 \times 4 \times 3 \times 2 \times 1} = \frac{2 + 6}{120} = \frac{1}{15}$$

i) $\frac{2! + 3! - 5!}{3! - 2!}$

Solution:

$$\frac{2 + 6 - 120}{6 - 2} = \frac{-112}{4} = -28$$

j) $(3!)^3$

Solution:

$6 \times 6 \times 6 = 216$

k) $\frac{3! \times 4!}{2!}$

Solution:

$$\frac{(3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)}{2 \times 1} = 72$$

2. Calculate the following using a calculator:

a) $\frac{12!}{2!}$

Solution:

$$\frac{(12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)}{2 \times 1} = 239\ 500\ 800$$

b) $\frac{10!}{20!}$

Solution:

$1{,}49 \times 10^{-12}$

c) $\frac{10! + 12!}{5! + 6!}$

Solution:

$574\ 560$

d) $5!(2! + 3!)$

Solution:

960

e) $(4!)^2(3!)^2$

Solution:

20 736

3. Show that the following is true:

a) $\dfrac{n!}{(n-2)!} = n^2 - n$

Solution:

$$\frac{n!}{(n-2)!} = \frac{n \times (n-1) \times \cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times 3 \times 2 \times 1}{\cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times 3 \times 2 \times 1}$$
$$= n(n-1) = n^2 - n$$

b) $\dfrac{(n-1)!}{n!} = \dfrac{1}{n}$

Solution:

$$\frac{(n-1)!}{n!} = \frac{\cancel{(n-1)} \times \cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times 3 \times 2 \times 1}{n \times \cancel{(n-1)} \times \cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times 3 \times 2 \times 1} = \frac{1}{n}$$

c) $\dfrac{(n-2)!}{(n-1)!} = \dfrac{1}{n-1}$ for $n > 1$

Solution:

$$\frac{(n-2)!}{(n-1)!} = \frac{\cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times 3 \times 2 \times 1}{(n-1) \times \cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times 3 \times 2 \times 1} = \frac{1}{n-1}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 29H4	1b. 29H5	1c. 29H6	1d. 29H7	1e. 29H8	1f. 29H9
1g. 29HB	1h. 29HC	1i. 29HD	1j. 29HF	1k. 29HG	2a. 29HH
2b. 29HJ	2c. 29HK	2d. 29HM	2e. 29HN	3a. 29HP	3b. 29HQ
3c. 29HR					

www.everythingmaths.co.za   m.everythingmaths.co.za

10.6 Application to counting problems

Exercise 10 – 6: Number of choices in a row

1. How many different possible outcomes are there for a swimming event with six competitors?

   Solution:

   $6! = 720$

2. How many different possible outcomes are there for the gold (1st), silver (2nd) and bronze (3rd) medals in a swimming event with six competitors?

   Solution:

   $6 \times 5 \times 4 = 120$

3. Susan wants to visit her friends in Pretoria, Johannesburg, Phalaborwa, East London and Port Elizabeth. In how many different ways can the visits be arranged?

   Solution:

   $5! = 120$ ways

4. A head boy, a deputy head boy, a head girl and a deputy head girl must be chosen out of a student council consisting of 18 girls and 18 boys. In how many ways can they be chosen?

   Solution:

   $18 \times 17 + 18 \times 17 = 612$ ways

5. Twenty different people enter a golf competition. Only the first six of them can win prizes. In how many different ways can the prizes be won?

   Solution:

   $20 \times 19 \times 18 \times 17 \times 16 \times 15 = 27\ 907\ 200$ ways

6. Three letters of the word 'EMPTY' are arranged in a row. How many different arrangements are possible?

   Solution:

   $5 \times 4 \times 3 = 60$ arrangements

7. Pool balls are numbered from 1 to 15. You have only one set of pool balls. In how many different ways can you arrange:

   a) all 15 balls. Write your answer in scientific notation, rounding off to two decimal places.

      Solution:

      $15! = 1{,}31 \times 10^{12}$

   b) four of the 15 balls.

      Solution:

      $15 \times 14 \times 13 \times 12 = 32\ 760$

8. The captains of all the sports teams in a school have to stand next to each other for a photograph. The school sports programme offers rugby, cricket, hockey, soccer, netball and tennis.

   a) In how many different orders can they stand in the photograph?

      Solution:

      $$6! = 720 \text{ different orders}$$

   b) In how many different orders can they stand in the photograph if the rugby captain stands on the extreme left and the cricket captain stands on the extreme right?

      Solution:

      Since we have no choice about where to put the rugby and cricket captains, there are only 4 people left to arrange.

      $$4! = 24 \text{ different orders}$$

   c) In how many different orders can they stand if the rugby captain, netball captain and cricket captain must stand next to each other?

      Solution:

      The rugby captain, netball captain and cricket captain are treated as a single object, as they must stand together. So there are four different objects to arrange therefore there are $4!$ different arrangements. The rugby captain, netball captain and cricket captain can also swop positions between themselves in $3!$ different ways, therefore there are:

      $$4! \times 3! = 144 \text{ different orders}$$

9. How many three-digit numbers can be made from the digits 1 to 6 if:

   a) repetition is not allowed?

      Solution:

      $$6 \times 5 \times 4 = 120$$

   b) repetition is allowed?

Solution:

$$6^3 = 216$$

10. There are two different red books and three different blue books on a shelf.

    a) In how many different ways can these books be arranged?

    Solution:

    $$5! = 120 \text{ different ways to arrange the books}$$

    b) If you want the red books to be together, in how many different ways can the books be arranged?

    Solution:

    If the red books are treated as a single object, there are four different objects to arrange therefore there are $4!$ different arrangements. The red books can also be rearranged between themselves in $2!$ different ways, therefore there are:

    $$4! \times 2! = 48 \text{ different ways to arrange the books}$$

    c) If you want all the red books to be together and all the blue books to be together, in how many different ways can the books be arranged?

    Solution:

    There are two groups of books, red and blue, which can be arranged $2!$ ways. Then there are two red books which can be arranged in $2!$ ways and there are three blue books which can be arranged in $3!$ ways. Therefore, there are

    $$2! \times 2! \times 3! = 24 \text{ different ways to arrange the books}$$

11. There are two different Mathematics books, three different Natural Sciences books, two different Life Sciences books and four different Accounting books on a shelf. In how many different ways can they be arranged if:

    a) the order does not matter?

    Solution:

    $$11! = 39\ 916\ 800 \text{ ways to arrange the books}$$

    b) all the books of the same subject stand together?

    Solution:

    There are four groups of books, which can be arranged in $4!$ different ways. Of the books, two are Mathematics books, three are Natural Sciences books, two are Life Sciences books and four are Accounting books. Therefore, there are:

    $$4! \times 2! \times 3! \times 2! \times 4! = 13\ 824 \text{ ways to arrange the books}$$

    c) the two Mathematics books stand first?

    Solution:

    The Mathematics books can be arranged in $2!$ ways while the remaining books can be arranged in $9!$ ways. Therefore, there are:

    $$2! \times 9! = 725\ 760 \text{ ways to arrange the books}$$

    d) the Accounting books stand next to each other?

    Solution:

    The Accounting books can be arranged in $4!$ ways and, if treated as a single object, can be arranged with the remaining books in $8!$ ways. Therefore, there are:

    $$4! \times 8! = 967\ 680$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29HS 2. 29HT 3. 29HV 4. 29HW 5. 29HX 6. 29HY
7. 29HZ 8. 29J2 9. 29J3 10. 29J4 11. 29J5

www.everythingmaths.co.za **m.everythingmaths.co.za**

Exercise 10 – 7: Number of arrangements of sets containing alike objects

1. You have the word 'EXCELLENT'.

    a) If the repeated letters are regarded as different letters, how many letter arrangements are possible?

    Solution:

    $$9! = 362\ 880$$

    b) If the repeated letters are regarded as identical, how many letter arrangements are possible?

    Solution:

    There are 3 E's and 2 L's so we have to divide by $3!$ and $2!$.

    $$\frac{9!}{3! \times 2!} = 30\ 240$$

    c) If the first and last letters are identical, how many letter arrangements are there?

    Solution:

    The word could start and end in E or L. With an E, there are $\frac{7!}{2!}$ letter arrangements (divide by 2 L's) and with an L there are $\frac{7!}{3!}$ letter arrangements (divide by 3 E's). Therefore, there are:

    $$\frac{7!}{3!} + \frac{7!}{2!} = 840 + 2520 = 3360 \text{ possible letter arrangements}$$

    d) How many letter arrangements can be made if the arrangement starts with an L?

    Solution:

    This is equivalent to removing one L from the letters available for arrangement. Therefore, there are:

    $$\frac{8!}{3!} = 6720 \text{ possible letter arrangements}$$

    e) How many letter arrangements are possible if the word ends in a T?

    Solution:

    This is equivalent to removing the T from the letters available for arrangement. Therefore, there are:

    $$\frac{8!}{3! \times 2!} = 3360 \text{ possible letter arrangements}$$

2. You have the word 'ASSESSMENT'.

    a) If the repeated letters are regarded as different letters, how many letter arrangements are possible?

    Solution:

    $$10! = 3\ 628\ 800$$

    b) If the repeated letters are regarded as identical, how many letter arrangements are possible?

    Solution:

    $$\frac{10!}{4! \times 2!} = 75\ 600$$

    c) If the first and last letters are identical, how many letter arrangements are there?

    Solution:

    The word could start and end in S or E. With an S, there are $\frac{8!}{2! \times 2!}$ letter arrangements (divide by 2 E's and 2 remaining S's) and with an E there are $\frac{8!}{4!}$ letter arrangements (divide by 4 S's). Therefore, there are:

    $$\frac{8!}{2! \times 2!} + \frac{8!}{4!} = 10\ 080\ + 1680 = 11\ 760 \text{ possible letter arrangements}$$

d) How many letter arrangements can be made if the arrangement starts with a vowel?

Solution:

The word could start with A or E. With an A, there are $\frac{9!}{4! \times 2!}$ letter arrangements (divide by 2 E's and 4 S's) and with an E there are $\frac{9!}{4!}$ letter arrangements (divide by 4 S's). Therefore, there are:

$$\frac{9!}{4! \times 2!} + \frac{9!}{4!} = 7560 + 15\,120 = 22\,680 \text{ possible letter arrangements}$$

e) How many letter arrangements are possible if all the S's are at the beginning of the word?

Solution:

This is equivalent to removing all the S's from the letters available for arrangement. Therefore, there are:

$$\frac{6!}{2!} = 360 \text{ possible letter arrangements}$$

3. On a piano the white keys represent the following notes: C, D, E, F, G, A, B. How many tunes, seven notes in length, can be composed with these notes if:

a) a note can be played only once?

Solution:

$$7! = 5040 \text{ possible tunes}$$

b) the notes can be repeated?

Solution:

$$7^7 = 823\,543 \text{ possible tunes}$$

c) the notes can be repeated and the tune begins and ends with a D?

Solution:

The tune starting and ending with a D leaves five possible positions in which to arrange the seven notes. Therefore, there are:

$$7^5 = 16\,807 \text{ possible tunes}$$

d) the tune consists of 3 D's, 2 B's and 2 A's.

Solution:

$$\frac{7!}{3! \times 2! \times 2!} = 210 \text{ possible tunes}$$

4. There are three black beads and four white beads in a row. In how many ways can the beads be arranged if:

a) same-coloured beads are treated as different beads?

Solution:

$$7! = 5040 \text{ ways}$$

b) same-coloured beads are treated as identical beads?

Solution:

$$\frac{7!}{3! \times 4!} = 35 \text{ ways}$$

5. There are eight balls on a table. Some are white and some are red. The white balls are all identical and the red balls are all identical. The balls are removed one at a time. In how many different orders can the balls be removed if:

a) seven of the balls are red?

Solution:

$$\frac{8!}{7!} = 8 \text{ different orders}$$

b) three of the balls are red?

Solution:

$$\frac{8!}{3! \times 5!} = 56 \text{ different orders}$$

c) there are four of each colour?

Solution:

$$\frac{8!}{4! \times 4!} = 70 \text{ different orders}$$

6. How many four-digit numbers can be formed with the digits 3, 4, 6 and 7 if:

a) there can be repetition?

Solution:

$$4^4 = 256 \text{ possible numbers}$$

b) each digit can only be used once?

Solution:

$$4! = 24 \text{ possible numbers}$$

c) if the number is odd and repetition is allowed?

Solution:

For the number to be odd, it must end in 3 or 7. For the numbers ending in 3, there are 4^3 different arrangements of the first three digits and similarly, for 7, there are 4^3 different arrangements. Therefore, there are

$$4^3 + 4^3 = 128 \text{ possible numbers which match the criteria}$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29J6 2. 29J7 3. 29J8 4. 29J9 5. 29JB 6. 29JC

www.everythingmaths.co.za m.everythingmaths.co.za

10.7 Application to probability problems

Exercise 10 – 8: Solving probability problems using the fundamental counting principle

1. A music group plans a concert tour in South Africa. They will perform in Cape Town, Port Elizabeth, Pretoria, Johannesburg, Bloemfontein, Durban and East London.

a) In how many different orders can they plan their tour if there are no restrictions?

Solution:

$$7! = 5040 \text{ different orders are possible}$$

b) In how many different orders can they plan their tour if their tour begins in Cape Town and ends in Durban?

Solution:

This reduces the available objects (cities) by two, hence

$$5! = 120 \text{ different orders are possible}$$

c) If the tour cities are chosen at random, what is the probability that their performances in Cape Town, Port Elizabeth, Durban and East London happen consecutively? Give your answer correct to 3 decimal places.

Solution:

If these cities are grouped together, they can be treated as a single object in the arrangement, hence there are $4!$ different ways to order the objects. Within the grouped cities, there are $4!$ different ways to order them. Therefore, there are $4! \times 4! = 576$ different orders

Therefore, the probability of any of the orders having Cape Town, Port Elizabeth, Durban and East London happen consecutively is:

$$\begin{aligned} P(\text{the 4 cities grouped}) &= \frac{n(\text{the 4 cities grouped})}{n(\text{total possible orders})} \\ &= \frac{576}{5040} = 0{,}114 \end{aligned}$$

2. A certain restaurant has the following course options available for a three-course set menu:

STARTERS	MAINS	DESSERTS
Calamari salad	Fried chicken	Ice cream and chocolate sauce
Oysters	Crumbed lamb chops	Strawberries and cream
Fish in garlic sauce	Mutton Bobotie	Malva pudding with custard
	Chicken schnitzel	Pears in brandy sauce
	Vegetable lasagne	
	Chicken nuggets	

a) How many different set menus are possible?

Solution:

$$3 \times 6 \times 4 = 72 \text{ different set menus}$$

b) What is the probability that a set menu includes a chicken course?

Solution:

$$\begin{aligned} n(\text{set menu with chicken}) &= 3 \times 3 \times 4 = 36 \\ n(\text{total set menus}) &= 72 \\ \text{Therefore } P(\text{set menu with chicken}) &= \frac{36}{72} = 0{,}5 \end{aligned}$$

3. Eight different pairs of jeans and 5 different shirts hang on a rail.

a) In how many different ways can the clothes be arranged on the rail?

Solution:

$$13! = 6\ 227\ 020\ 800 \text{ different ways}$$

b) In how many ways can the clothing be arranged if all the jeans hang together and all the shirts hang together?

Solution:

The shirts and jeans form two groups, which can be arranged $2!$ ways. The five shirts can be arranged $5!$ ways and the eight pairs of jeans can be arranged $8!$ ways.

$$2! \times 8! \times 5! = 9\ 676\ 800$$

c) What is the probability, correct to three decimal places, of the clothing being arranged on the rail with a shirt at one end and a pair of jeans at the other?

Solution:

- The five different choices of shirt and eight different choices of pairs of jeans can form 5×8 different arrangements at the ends of the rail.
- There are $2!$ different ways to arrange a shirt at one end and a pair of jeans on the other: S - - - - - - - - - - - J and J - - - - - - - - - - - S

- If a shirt is at one end and a pair of jeans at the other, there remains $11!$ different arrangements of the remaining clothing items.

Therefore, there are:

$2 \times 8 \times 5 \times 11! = 3\ 193\ 344\ 000$ different ways to arrange the clothing

The probability of a clothing arrangement with a shirt at one end and a pair of jeans at the other $= \frac{3\ 193\ 344\ 000}{6\ 227\ 020\ 800} = 0{,}513$

4. A photographer places eight chairs in a row in his studio in order to take a photograph of the debating team. The team consists of three boys and five girls.

   a) In how many ways can the debating team be seated?

   Solution:

$$8! = 40\ 320$$

   b) What is the probability that a particular boy and a particular girl sit next to each other?

   Solution:

   Regard the particular boy and girl as one group. The group can be seated $2!$ ways. The number of ways that this group and the remaining 6 people can sit $= 7!$. Therefore the total number of ways this particular boy and girl can sit together in the photograph $= 2! \times 7! = 10\ 080$. The probability of a particular boy and girl sitting together $= \frac{10\ 080}{40\ 320} = 0{,}25$

5. If the letters of the word 'COMMITTEE' are randomly arranged, what is the probability that the letter arrangements start and end with the same letter?

   Solution:

   - There are 2 M's, 2 T's and 2 E's and a total of 9 letters.
   - Total number of letter arrangements $= \dfrac{9!}{2! \times 2! \times 2!} = 45\ 360$
   - Possibilities of the first and last letter being the same:
     - M(COITTEE)M
       Total of 7 letters of which there are 2E's and 2T's
       Number of letter arrangements $= \dfrac{7!}{2! \times 2!} = 1260$
     - T(MCOIEEM)T
       Total of 7 letters of which there are 2E's and 2M's
       Number of letter arrangements $= \dfrac{7!}{2! \times 2!} = 1260$
     - E(TTMCOIM)E
       Total of 7 letters of which there are 2T's and 2M's
       Number of letter arrangements $= \dfrac{7!}{2! \times 2!} = 1260$
   - Total number of letter arrangements if the letter arrangement starts and ends with the same letter $= 3 \times 1260 = 3780$.

$$P(\text{first and last letter the same}) = \frac{3780}{45\ 360} = \frac{1}{12}$$

6. Four different Mathematics books, three different Economics books and two different Geography books are arranged on a shelf. What is the probability that all the books of the same subject are arranged next to each other?

   Solution:

   Total number of different ways the books can be arranged $= 9! = 362\ 880$.There are 3 subjects of books which can be arranged $3!$ ways.

   Therefore, the total number of arrangements if the subjects are arranged together $= 3! \times 4! \times 3! \times 2! = 1728$

$$P(\text{books of the same subject next to each other}) = \frac{1728}{362\ 880} = \frac{1}{210}$$

7. A number plate is made up of three letters of the alphabet (excluding F and S) followed by three digits from 0 to 9. The numbers and letters can be repeated. Calculate the probability that a randomly chosen number plate:

a) starts with the letter D and ends with the digit 3.

Solution:

Total number of arrangements $= 24^3 \times 10^3$

Total number of arrangements beginning with D and ending with $3 = 24^2 \times 10^2$

$$P(\text{first D and last } 3) = \frac{24^2 \times 10^2}{24^3 \times 10^3} = \frac{1}{240}$$

b) has precisely one D.

Solution:

The 'D' could be at the first, second or third positions. The other two letters cannot include a 'D', leaving 23 other letters. Therefore there are

$$23^2 \times 10^3 \times 3 \text{ different arrangements containing only one D}$$

$$\text{Therefore } P(\text{only one D}) = \frac{23^2 \times 10^3 \times 3}{24^3 \times 10^3} = \frac{529}{4608}$$

c) contains at least one 5.

Solution:

$$\begin{aligned} P(\text{contains at least one } 5) &= 1 - P(\text{no 5s}) \\ &= 1 - \frac{24^3 \times 9^3}{24^3 \times 10^3} \\ &= 1 - 0{,}729 = 0{,}271 \end{aligned}$$

8. In the 13-digit identification (ID) numbers of South African citizens:

- The first six numbers are the birth date of the person in YYMMDD format.
- The next four digits indicate gender, with 5000 and above being male and 0001 to 4999 being female.
- The next number is the country ID; 0 is South Africa and 1 is not.
- The second last number used to be a racial identifier but it is now 8 for everybody.
- The last number is a control digit, which verifies the rest of the number.

Assume that the control digit is a randomly generated digit from 0 to 9 and ignore the fact that leap years have an extra day.

a) Calculate the total number of possible ID numbers.

Solution:

For all available arrangements, if an ID number is structured ABCDEFGHIJKLM:

- A and B are any digits between 00 and 99 (year)
- C and D are any digits from 01 to 12 (month)
- E and F are any digits from 01 to 28, 30 or 31 dependent on month (day)
- G, H, I and J are any digits from 0001 to 9999 (gender)
- K is either a 0 or 1
- L is an 8
- M is any digit between 0 and 9

Therefore, the total number of possible ID numbers for 30-day months is:

$$100 \times 4 \times 30 \times 9999 \times 2 \times 1 \times 10 = 2\,399\,760\,000$$

Therefore, the total number of possible ID numbers for 31-day months is:

$$100 \times 7 \times 31 \times 9999 \times 2 \times 1 \times 10 = 4\,339\,566\,000$$

Therefore, the total number of possible ID numbers for February is:

$$100 \times 1 \times 28 \times 9999 \times 2 \times 1 \times 10 = 559\,944\,000$$

Therefore, the total number of possible ID numbers is:

$$2\,399\,760\,000 + 4\,339\,566\,000 + 559\,944\,000 = 9\,239\,076\,000$$

b) Calculate the probability that a randomly generated ID number is of a South African male born during the 1980s. Write your answer correct to two decimal places.

Solution:

There is a lot of information in the problem and we can simplify it by identifying the relevant information. We want to calculate the probability that an ID number is for a South African male born during the 1980s. This means that we have to look at the digits for country, gender and year of birth.

If an ID number is structured ABCDEFGHIJKLM:

- For a South African, K is 0.
- For a male, GHIJ are from 5000 to 9999.
- For someone born during the 1980s, AB are between 80 and 89.

This gives $1 \times 5000 \times 10 = 50\ 000$ combinations for ABGHIJK

Without any restrictions, the total combinations for ABGHIJK is $2 \times 9999 \times 100 = 1\ 999\ 800$.

$$\text{Therefore } P(\text{SA male 80s}) = \frac{50\ 000}{1\ 999\ 800} = 0{,}025$$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29JD 2. 29JF 3. 29JG 4. 29JH 5. 29JJ 6. 29JK
7. 29JM 8. 29JN

www.everythingmaths.co.za m.everythingmaths.co.za

10.8 Summary

Exercise 10 – 9: End of chapter exercises

1. An ATM card has a four-digit PIN. The four digits can be repeated and each of them can be chosen from the digits 0 to 9.

   a) What is the total number of possible PINs?

   Solution:

   $$10^4 = 10\ 000$$

   b) What is the probability of guessing the first digit correctly?

   Solution:

   $\frac{1}{10}$

   c) What is the probability of guessing the second digit correctly?

   Solution:

   $\frac{1}{10}$

   d) If your ATM card is stolen, what is the probability, correct to four decimal places, of a thief guessing all four digits correctly on his first guess?

   Solution:

   $\left(\frac{1}{10}\right)^4 = 0{,}0001$

   e) After three incorrect PIN attempts, an ATM card is blocked from being used. If your ATM card is stolen, what is the probability, correct to four decimal places, of a thief blocking the card? Assume the thief enters a different PIN each time.

   Solution:

$$
\begin{aligned}
P(\text{incorrect PIN}) &= 1 - P(\text{correct PIN}) \\
&= 1 - \frac{1}{10^4} = 0{,}9999 \\
\text{Therefore } P(3 \text{ incorrect PIN attempts}) &= (0{,}9999)^3 \\
&= 0{,}9997
\end{aligned}
$$

2. The LOTTO rules state the following:

   - Six numbers are drawn from the numbers 1 to 49 - this is called a 'draw'.
   - Numbers are not replaced once drawn, so you cannot have the same number more than once.
   - The order of the drawn numbers does not matter.

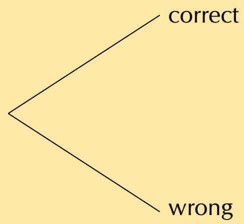
   You decide to buy one LOTTO ticket consisting of 6 numbers.

   a) How many different possible LOTTO draws are there? Write your answer in scientific notation, rounding to two digits after the decimal point.

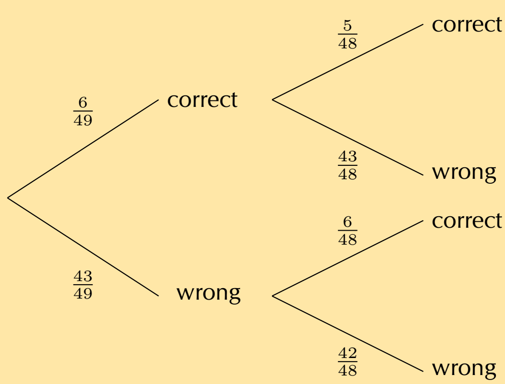
   Solution:

   $$49 \times 48 \times 47 \times 46 \times 45 \times 44 = 1{,}01 \times 10^{10}$$

   b) Complete the tree diagram below after the first two LOTTO numbers have been drawn showing the possible outcomes and probabilities of the numbers on your ticket.



   Solution:



   c) What is the probability of getting the first number drawn correctly?

   Solution:

   $$P(\text{first number correct}) = \frac{6}{49}$$

   d) What is the probability of getting the second number drawn correctly if you get the first number correct?

   Solution:

   $$P(\text{second number correct if first correct}) = \frac{5}{48}$$

   e) What is the probability of getting the second number drawn correct if you do not get the first number correctly?

Solution:

$$P(\text{second number correct if first incorrect}) = \frac{6}{48}$$

f) What is the probability of getting the second number drawn correct?

Solution:

This is the sum of the probabilities of the outcomes where the second number drawn is correct:

$$\begin{aligned} P(\text{second number correct}) &= \frac{6}{49} \times \frac{5}{48} + \frac{43}{49} \times \frac{6}{48} \\ &= \frac{5}{392} + \frac{43}{392} \\ &= \frac{6}{49} \end{aligned}$$

Notice that the answer is the same as the probability of getting the first number correct. If you are unaware of the outcome of prior events, the probability of the outcome of a certain event, is equal to the probability of that outcome of the first event. Learners are not required to learn this concept but it is interesting to note.

g) What is the probability of getting all 6 LOTTO numbers correct? Write your answer in scientific notation, rounding to two digits after the decimal point.

Solution:

$$P(\text{all 6 correct}) = \frac{6}{49} \times \frac{5}{48} \times \frac{4}{47} \times \frac{3}{46} \times \frac{2}{45} \times \frac{1}{44} = 7{,}15 \times 10^{-8}$$

3. The population statistics of South Africa show that 55% of all babies born are female. Calculate the probability that a couple planning to have children will have a boy followed by a girl and then a boy. Assume that each birth is an independent event. Write your answer as a percentage, correct to two decimal places.

   Solution:

   $$0{,}45 \times 0{,}55 \times 0{,}45 = 11{,}14\%$$

4. Fezile and Vuzi write a Mathematics test. The probability that Fezile will pass the test is $0{,}8$. The probability that Vuzi will pass the test is $0{,}75$. What is the probability that only one of them will pass the test?

   Solution:

   $$\begin{aligned} P(\text{only one passes}) &= P(\text{F pass}) \times P(\text{V fail}) + P(\text{F fail}) \times P(\text{V pass}) \\ &= 0{,}8 \times 0{,}25 + 0{,}2 \times 0{,}75 \\ &= 0{,}35 \end{aligned}$$

5. Landline telephone numbers are 10 digits long. Numbers begin with a zero followed by 9 digits chosen from the digits 0 to 9. Repetitions are allowed.

   a) How many different phone numbers are possible?

   Solution:

   $$10^9 = 1\ 000\ 000\ 000$$

   b) The first three digits of a number form an area code. The area code for Cape Town is 021. How many different phone numbers are available in the Cape Town area?

   Solution:

   $$10^7 = 10\ 000\ 000$$

   c) What is the probability of the second digit being an even number?

Solution:

There are 5 even numbers between 0 and 9, therefore:

$$P(\text{second digit even}) = \frac{5}{10} = \frac{1}{2}$$

d) Ignoring the first digit, what is the probability of a phone number consisting of only odd digits? Write your answer correct to three decimal places.

Solution:

$$\left(\frac{5}{10}\right)^9 = 0{,}002$$

6. Take the word 'POSSIBILITY'.

a) In how many way can the letters be arranged if repeated letters are considered identical?

Solution:

There are two S's and three I's, therefore there are:

$$\frac{11!}{2! \times 3!} = 3\ 326\ 400 \text{ different arrangements}$$

b) What is the probability that a randomly generated arrangement of the letters will begin with three I's? Write your answer as a fraction.

Solution:

If the arrangement begins with three I's, that leaves POSSBLTY to be arranged.

$$\begin{aligned} n(\text{arrangements beginning with III}) &= \frac{8!}{2!} = 20\ 160 \\ \text{Therefore } P(\text{arrangements beginning with III}) &= \frac{20\ 160}{3\ 326\ 400} \\ &= \frac{1}{165} \end{aligned}$$

7. The code to a safe consists of 10 digits chosen from the digits 0 to 9. None of the digits are repeated. Determine the probability of a code where the first digit is odd and none of the first three digits may be a zero. Write your answer as a percentage, correct to two decimal places.

Solution:

- There are 5 odd numbers from 0 to 9, so there are five arrangements for the first digit.
- For the second digit there are 8 digits left to arrange, as the first digit and zero are removed.
- For the third digit there are 7 digits left to arrange as zero and the first two digits are removed.
- For the fourth digit, there are 7 digits (0 is now included) left which can be combined in $7!$ ways for the remaining digits.

$$\begin{aligned} \text{Total possible number of codes} &= 10! = 3\ 628\ 800 \\ \text{Total no. of codes with first digit odd, first three non-zero} &= 5 \times 8 \times 7 \times 7! \\ &= 1\ 411\ 200 \\ \text{Therefore } P(\text{first digit odd, first three non-zero}) &= \frac{1\ 411\ 200}{3\ 628\ 800} \\ &= 38{,}89\% \end{aligned}$$

8. Four different red books and three different blue books are to be arranged on a shelf. What is the probability that all the red books and all the blue books stand together on the shelf?

Solution:

Total number of arrangements $= 7!$

If the red and blue books stand together, there are two groups of books which can be arranged $2!$ ways.

Then there are four red books which can be arranged in $4!$ ways and the three blue books can be arranged in $3!$ ways.

$$P(\text{blue and red together}) = \frac{2! \times 3! \times 4!}{7!} = \frac{2}{35}$$

9. The probability that Thandiswa will go dancing on a Saturday night (event D) is $0{,}6$ and the probability that she will go watch a movie is $0{,}3$ (event M). Determine the probability that she will:

   a) go dancing and watch a movie if D and M are independent.

   Solution:

$$\begin{aligned} P(D \text{ and } M) &= P(D) \times P(M) \\ &= 0{,}6 \times 0{,}3 = 0{,}18 \end{aligned}$$

   b) go dancing or watch a movie if D and M are mutually exclusive.

   Solution:

$$\begin{aligned} P(D \text{ or } M) &= P(D) + P(M) \\ &= 0{,}6 + 0{,}3 = 0{,}9 \end{aligned}$$

   c) go dancing and watch a movie if $P(D \text{ or } M) = 0{,}7$.

   Solution:

$$\begin{aligned} P(D \text{ and } M) &= P(D) + P(M) - P(D \text{ or } M) \\ &= 0{,}6 + 0{,}3 - 0{,}7 = 0{,}2 \end{aligned}$$

   d) not go dancing or go to a movie if $P(D \text{ and } M) = 0{,}8$.

   Solution:

$$\begin{aligned} P(\text{not } (D \text{ and } M)) &= 1 - P(D \text{ and } M) \\ &= 1 - 0{,}8 = 0{,}2 \end{aligned}$$

10. Three boys and four girls sit in a row.

   a) In how many ways can they sit in the row?

   Solution:

$$7! = 5040$$

   b) What is the probability that they sit in alternating gender positions?

   Solution:

   There is only one way they can sit alternately: GBGBGBG

   The number of ways they can sit alternately $= 1! \times 3! \times 4!$

$$P(\text{sit in alternating positions}) = \frac{1! \times 3! \times 4!}{7!} = \frac{1}{35}$$

11. The number plate on a car consists of any 3 letters of the alphabet (excluding the vowels, J and Q), followed by any 3 digits from 0 to 9. For a car chosen at random, what is the probability that the number plate starts with a Y and ends with an odd digit? Write your answer as a fraction.

   Solution:

   - The number plate starts with a Y, so there is only 1 choice for the first letter.

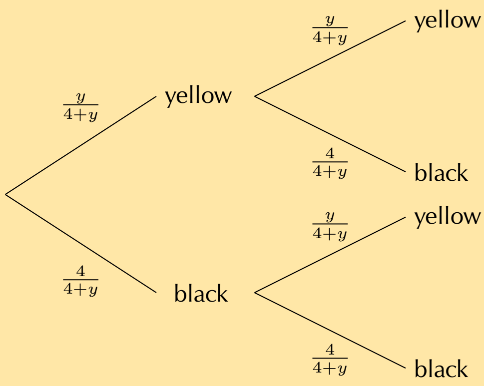
- The number plate ends with an odd digit, so there are 5 choices (1, 3, 5, 7, 9)
- There are 19 letters available because the 5 vowels (A, E, I, O, U), J and Q are excluded.

$$\begin{aligned} n(\text{plates starting with Y, ending with odd digit}) &= 1 \times 19^2 \times 10^2 \times 5 \\ &= 180\ 500 \\ n(\text{total possible number plates} &= 19^3 \times 10^3 \\ &= 6\ 859\ 000 \\ \therefore P(\text{plate starting with Y, ending with odd digit}) &= \frac{180\ 500}{6\ 859\ 000} \\ &= \frac{1}{38} \end{aligned}$$

12. There are four black balls and y yellow balls in a bag. Thandi takes out a ball, notes its colour and then puts it back in the bag. She then takes out another ball and also notes its colour. If the probability that both balls have the same colour is $\frac{5}{8}$, determine the value of y.

Solution:

Using a tree diagram, the different outcomes and probabilities can be illustrated as follows:



Solving for y:

$$\begin{aligned} \frac{5}{8} &= \frac{y}{4+y} \times \frac{y}{4+y} + \frac{4}{4+y} \times \frac{4}{4+y} \\ \frac{5}{8} &= \left(\frac{y}{4+y}\right)^2 + \left(\frac{4}{4+y}\right)^2 \\ \frac{5}{8} &= \frac{y^2+4^2}{(4+y)^2} \\ 5(4+y)^2 &= 8(y^2+16)) \\ 5(16+8y+y^2) &= 8y^2+128 \\ 80+40y+5y^2 &= 8y^2+128 \\ \text{Therefore } 3y^2-40y+48 &= 0 \\ (3y-4)(y-12) &= 0 \\ \text{Therefore } y &= 12 (y \neq \frac{4}{3} \text{ as balls cannot be fractions}) \end{aligned}$$

13. A rare kidney disease affects only 1 in 1000 people and the test for this disease has a 99% accuracy rate.

    a) Draw a two-way contingency table showing the results if 100 000 of the general population are tested.

Solution:

If $100\ 000$ people are tested and the prevalence rate is $0{,}1\%$:

$$100\ 000 \times 0{,}001 = 100 \text{ people are expected to be sick}$$
$$\text{Therefore } 100\ 000 - 100 = 99\ 900 \text{ people are expected to be healthy}$$

	Sick	Healthy	Total
Positive			
Negative			
Total	100	99 900	100 000

If the test is 99% accurate:

$$100 \times 0{,}99 = 99 \text{ sick people are expected to test positive}$$
$$\therefore 100 - 99 = 1 \text{ sick person is expected to test negative}$$
$$\text{And } 99\ 900 \times 0{,}99 = 98\ 901 \text{ healthy people are expected to test negative}$$
$$\therefore 99\ 900 - 98\ 901 = 999 \text{ healthy people are expected to test positive}$$

	Sick	Healthy	Total
Positive	99	999	1098
Negative	1	98 901	98 902
Total	100	99 900	100 000

b) Calculate the probability that a person who tests positive for this rare kidney disease is sick with the disease, correct to two decimal places.

Solution:

$$\begin{aligned} P(\text{sick if tested positive}) &= \frac{n(\text{sick and tested positive})}{n(\text{tested positive})} \\ &= \frac{99}{1098} \\ &= 0{,}09 \end{aligned}$$

Notice that this means that a positive result is wrong 91% of the time! This is an important concept in medical science. For very rare diseases, tests have to be highly accurate otherwise the result is meaningless.

Check answers online with the exercise code below or click on 'show me the answer'.

1\. 29JP 2. 29JQ 3. 29JR 4. 29JS 5. 29JT 6. 29JV
7\. 29JW 8. 29JX 9. 29JY 10. 29JZ 11. 29K2 12. 29K3
13\. 29GT

www.everythingmaths.co.za m.everythingmaths.co.za