CHAPTER 1

Sequences and series

In earlier grades we learnt about number patterns, which included linear sequences with a common difference and quadratic sequences with a common second difference. We also looked at completing a sequence and how to determine the general term of a sequence.

In this chapter we also look at geometric sequences, which have a constant ratio between consecutive terms. We will learn about arithmetic and geometric series, which are the summing of the terms in sequences.

1.1 Arithmetic sequences EMCDP

An arithmetic sequence is a sequence where consecutive terms are calculated by adding a constant value (positive or negative) to the previous term. We call this constant value the common difference (d).

For example,

$$3; 0; -3; -6; -9; \ldots$$

This is an arithmetic sequence because we add -3 to each term to get the next term:

First term	T_1		3
Second term	T_2	$3 + (-3) =$	0
Third term	T_3	$0 + (-3) =$	-3
Fourth term	T_4	$-3 + (-3) =$	-6
Fifth term	T_5	$-6 + (-3) =$	-9
$\vdots$	$\vdots$	$\vdots$	$\vdots$

See video: 284G at www.everythingmaths.co.za

Exercise 1 – 1: Arithmetic sequences

Find the common difference and write down the next 3 terms of the sequence.

1. $2; 6; 10; 14; 18; 22; \ldots$
2. $-1; -4; -7; -10; -13; -16; \ldots$
3. $-5; -3; -1; 1; 3; \ldots$
4. $-1; 10; 21; 32; 43; 54; \ldots$
5. $a - 3b; a - b; a + b; a + 3b; \ldots$
6. $-2; -\frac{3}{2}; -1; -\frac{1}{2}; 0; \frac{1}{2}; 1; \ldots$
7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 284H 2. 284J 3. 284K 4. 284M 5. 284N 6. 284P

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The general term for an arithmetic sequence EMCDQ

For a general arithmetic sequence with first term a and a common difference d, we can generate the following terms:

$$
\begin{aligned}
T_1 &= a \\
T_2 &= T_1 + d = a + d \\
T_3 &= T_2 + d = (a + d) + d = a + 2d \\
T_4 &= T_3 + d = (a + 2d) + d = a + 3d \\
&\vdots \\
T_n &= T_{n-1} + d = (a + (n-2)d) + d = a + (n-1)\, d
\end{aligned}
$$

Therefore, the general formula for the n^{th} term of an arithmetic sequence is:

$$T_n = a + (n-1)\, d$$

DEFINITION: *Arithmetic sequence*

An arithmetic (or linear) sequence is an ordered set of numbers (called terms) in which each new term is calculated by adding a constant value to the previous term:

$$T_n = a + (n-1)d$$

where

- T_n is the n^{th} term;
- n is the position of the term in the sequence;
- a is the first term;
- d is the common difference.

Test for an arithmetic sequence

To test whether a sequence is an arithmetic sequence or not, check if the difference between any two consecutive terms is constant:

$$d = T_2 - T_1 = T_3 - T_2 = \ldots = T_n - T_{n-1}$$

If this is not true, then the sequence is not an arithmetic sequence.

Worked example 1: Arithmetic sequence

QUESTION

Given the sequence $-15; -11; -7; \ldots 173$.

1. Is this an arithmetic sequence?
2. Find the formula of the general term.
3. Determine the number of terms in the sequence.

SOLUTION

Step 1: Check if there is a common difference between successive terms

$$
\begin{aligned}
T_2 - T_1 &= -11 - (-15) = 4 \\
T_3 - T_2 &= -7 - (-11) = 4
\end{aligned}
$$

$\therefore$ This is an arithmetic sequence with $d = 4$

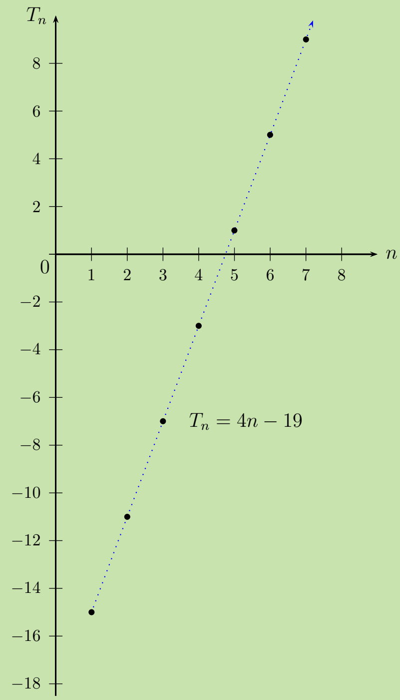
Step 2: Determine the formula for the general term

Write down the formula and the known values:

$$T_n = a + (n-1)d$$

$$a = -15; \qquad d = 4$$

$$
\begin{aligned}
T_n &= a + (n-1)d \\
&= -15 + (n-1)(4) \\
&= -15 + 4n - 4 \\
&= 4n - 19
\end{aligned}
$$



A graph was not required for this question but it has been included to show that the points of the arithmetic sequence lie in a straight line.

Note: The numbers of the sequence are natural numbers ($n \in \{1; 2; 3; \ldots\}$) and therefore we should not connect the plotted points. In the diagram above, a dotted line has been used to show that the graph of the sequence lies on a straight line.

Step 3: Determine the number of terms in the sequence

$$
\begin{aligned}
T_n &= a + (n-1)d \\
173 &= 4n - 19 \\
192 &= 4n \\
\therefore n &= \frac{192}{4} \\
&= 48 \\
\therefore T_{48} &= 173
\end{aligned}
$$

Step 4: Write the final answer

Therefore, there are 48 terms in the sequence.

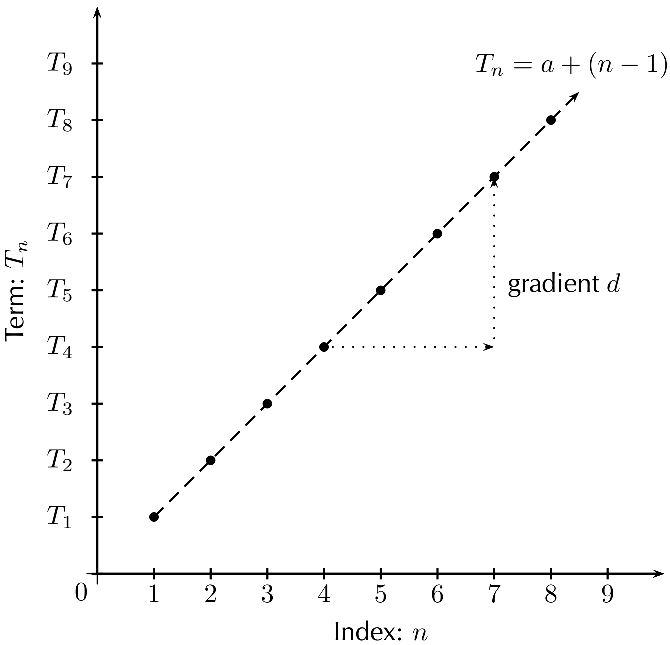
Arithmetic mean

The arithmetic mean between two numbers is the number half-way between the two numbers. In other words, it is the average of the two numbers. The arithmetic mean and the two terms form an arithmetic sequence.

For example, the arithmetic mean between 7 and 17 is calculated:

$$\begin{aligned}\text{Arithmetic mean } &= \frac{7+17}{2}\\ &= 12\\ \therefore 7; 12; 17 &\text{ is an arithmetic sequence}\\ T_2 - T_1 &= 12 - 7 = 5\\ T_3 - T_2 &= 17 - 12 = 5\end{aligned}$$

Plotting a graph of the terms of a sequence sometimes helps in determining the type of sequence involved. For an arithmetic sequence, plotting T_n vs. n results in the following graph:

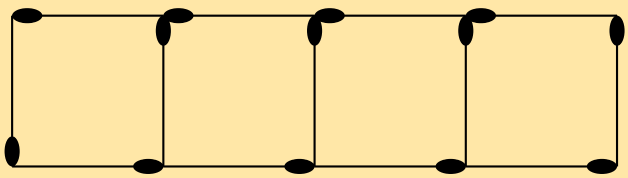


- If the sequence is arithmetic, the plotted points will lie in a straight line.
- Arithmetic sequences are also called linear sequences, where the common difference (d) is the gradient of the straight line.

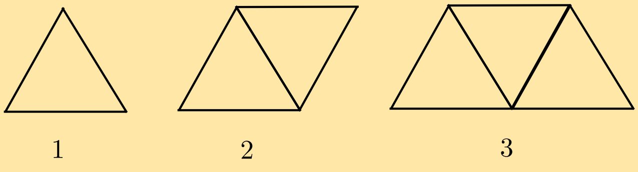
$$\begin{aligned}T_n &= a + (n-1)d\\ \text{can be written as } T_n &= d(n-1) + a\\ \text{which is of the same form as } y &= mx + c\end{aligned}$$

Exercise 1 – 2: Arithmetic Sequences

1. Given the sequence $7; 5{,}5; 4; 2{,}5; \ldots$
   a) Find the next term in the sequence.
   b) Determine the general term of the sequence.
   c) Which term has a value of -23?
2. Given the sequence $2; 6; 10; 14; \ldots$
   a) Is this an arithmetic sequence? Justify your answer by calculation.
   b) Calculate T_{55}.
   c) Which term has a value of 322?
   d) Determine by calculation whether or not 1204 is a term in the sequence?
3. An arithmetic sequence has the general term $T_n = -2n + 7$.
   a) Calculate the second, third and tenth terms of the sequence.
   b) Draw a diagram of the sequence for $0 < n \leq 10$.
4. The first term of an arithmetic sequence is $-\frac{1}{2}$ and $T_{22} = 10$. Find T_n.
5. What are the important characteristics of an arithmetic sequence?
6. You are given the first four terms of an arithmetic sequence. Describe the method you would use to find the formula for the n^{th} term of the sequence.
7. A single square is made from 4 matchsticks. To make two squares in a row takes 7 matchsticks, while three squares in a row takes 10 matchsticks.

   a) Write down the first four terms of the sequence.
   b) What is the common difference?
   c) Determine the formula for the general term.
   d) How many matchsticks are in a row of 25 squares?
   e) If there are 109 matchsticks, calculate the number of squares in the row.
8. A pattern of equilateral triangles decorates the border of a girl's skirt. Each triangle is made by three stitches, each having a length of 1 cm.

   a) Complete the table:

Figure no.	1	2	3	q	r	n
No. of stitches	3	5	p	15	71	s

   b) The border of the skirt is 2 m in length. If the entire length of the border is decorated with the triangular pattern, how many stitches will there be?

9. The terms $p; (2p+2); (5p+3)$ form an arithmetic sequence. Find p and the 15^{th} term of the sequence.
   [IEB, Nov 2011]
10. The arithmetic mean of $3a-2$ and x is $4a-4$. Determine the value of x in terms of a.
11. Insert seven arithmetic means between the terms $(3s-t)$ and $(-13s+7t)$.
12. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 284Q 2. 284R 3. 284S 4. 284T 5. 284V 6. 284W
7. 284X 8. 284Y 9. 284Z 10. 2852 11. 2853

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DEFINITION: *Quadratic sequence*

A quadratic sequence is a sequence of numbers in which the second difference between any two consecutive terms is constant.

The general formula for the n^{th} term of a quadratic sequence is:

$$T_n = an^2 + bn + c$$

	$n=1$		$n=2$		$n=3$		$n=4$
T_n	$a+b+c$		$4a+2b+c$		$9a+3b+c$		$16a+4b+c$
1st difference		$3a+b$		$5a+b$		$7a+b$	
2nd difference			$2a$		$2a$		

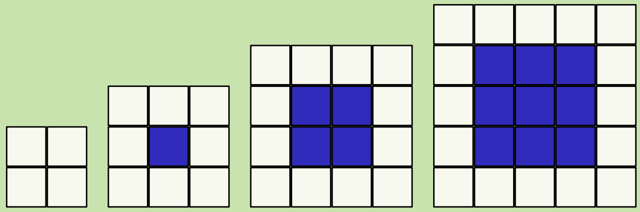
It is important to note that the first differences of a quadratic sequence form an arithmetic sequence. This sequence has a common difference of $2a$ between consecutive terms. In other words, a linear sequence results from taking the first differences of a quadratic sequence.

Worked example 2: Quadratic sequence

QUESTION

Consider the pattern of white and blue blocks in the diagram below.

1. Determine the sequence formed by the white blocks (w).
2. Find the sequence formed by the blue blocks (b).



Pattern number (n)	1	2	3	4	5	6	n
No. of white blocks (w)							
Common difference (d)							

Pattern number (n)	1	2	3	4	5	6	n
No. of blue blocks (b)							
Common difference (d)							

SOLUTION

Step 1: Use the diagram to complete the table for the white blocks

Pattern number (n)	1	2	3	4	5	6	n
No. of white blocks (w)	4	8	12	16	20	24	$4n$
Common difference (d)		4	4	4	4	4	

We see that the next term in the sequence is obtained by adding 4 to the previous term, therefore the sequence is linear and the common difference (d) is 4.

The general term is:

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 4 + (n-1)(4) \\ &= 4 + 4n - 4 \\ &= 4n \end{aligned}$$

Step 2: Use the diagram to complete the table for the blue blocks

Pattern number (n)	1	2	3	4	5	6
No. of blue blocks (b)	0	1	4	9	16	25
Difference		1	3	5	7	9

We notice that there is no common difference between successive terms. However, there is a pattern and on further investigation we see that this is in fact a quadratic sequence:

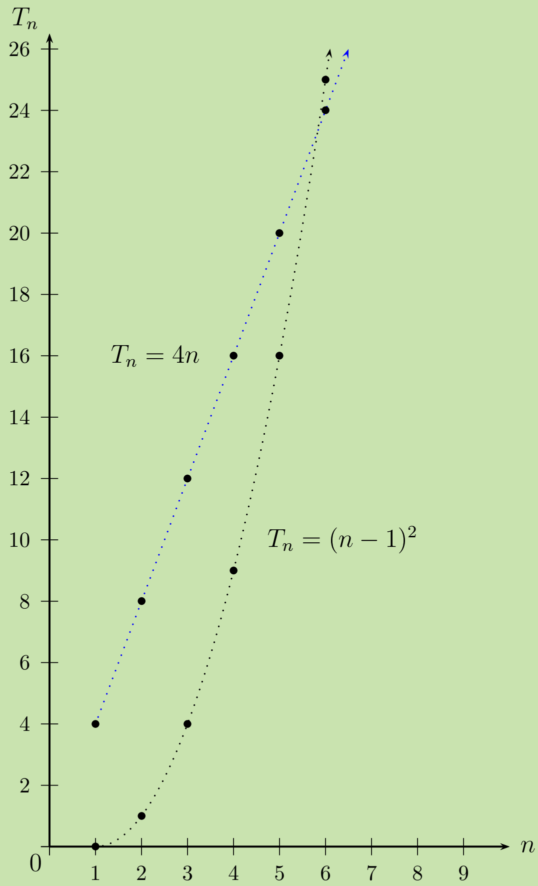
Pattern number (n)	1	2	3	4	5	6	n
No. of blue blocks (b)	0	1	4	9	16	25	$(n-1)^2$
First difference	–	1	3	5	7	9	–
Second difference	–	–	2	2	2	2	–
Pattern	$(1-1)^2$	$(2-1)^2$	$(3-1)^2$	$(4-1)^2$	$(5-1)^2$	$(6-1)^2$	$(n-1)^2$

$$T_n = (n-1)^2$$

Step 3: Draw a graph of T_n vs. n for each sequence

White blocks: $T_n = 4n$

Blue blocks: $T_n = (n-1)^2$
$= n^2 - 2n + 1$



Since the numbers of the sequences are natural numbers ($n \in \{1; 2; 3; \ldots\}$), we should not connect the plotted points. In the diagram above, a dotted line has been used to show that the graph of the sequence formed by the white blocks (w) is a straight line and the graph of the sequence formed by the blue blocks (b) is a parabola.

Exercise 1 – 3: Quadratic sequences

1. Determine whether each of the following sequences is:
   - a linear sequence;
   - a quadratic sequence;
   - or neither.

   a) $8; 17; 32; 53; 80; \ldots$
   b) $3p^2; 6p^2; 9p^2; 12p^2; 15p^2; \ldots$
   c) $1; 2{,}5; 5; 8{,}5; 13; \ldots$
   d) $2; 6; 10; 14; 18; \ldots$
   e) $5; 19; 41; 71; 109; \ldots$
   f) $3; 9; 16; 21; 27; \ldots$
   g) $2k; 8k; 18k; 32k; 50k; \ldots$
   h) $2\frac{1}{2}; 6; 10\frac{1}{2}; 16; 22\frac{1}{2}; \ldots$

2. A quadratic pattern is given by $T_n = n^2 + bn + c$. Find the values of b and c if the sequence starts with the following terms:

$$-1\ ;\ 2\ ;\ 7\ ;\ 14\ ;\ \ldots$$

3. $a^2; -a^2; -3a^2; -5a^2; \ldots$ are the first 4 terms of a sequence.

   a) Is the sequence linear or quadratic? Motivate your answer.
   b) What is the next term in the sequence?
   c) Calculate T_{100}.

4. Given $T_n = n^2 + bn + c$, determine the values of b and c if the sequence starts with the terms:

$$2\ ;\ 7\ ;\ 14\ ;\ 23\ ;\ \ldots$$

5. The first term of a quadratic sequence is 4, the third term is 34 and the common second difference is 10. Determine the first six terms in the sequence.
6. A quadratic sequence has a second term equal to 1, a third term equal to -6 and a fourth term equal to -14.

   a) Determine the second difference for this sequence.
   b) Hence, or otherwise, calculate the first term of the pattern.

7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 2854 1b. 2855 1c. 2856 1d. 2857 1e. 2858 1f. 2859
1g. 285B 1h. 285C 2. 285D 3. 285F 4. 285G 5. 285H
6. 285J

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1.2 Geometric sequences EMCDR

DEFINITION: *Geometric sequence*

A geometric sequence is a sequence of numbers in which each new term (except for the first term) is calculated by multiplying the previous term by a constant value called the constant ratio (r).

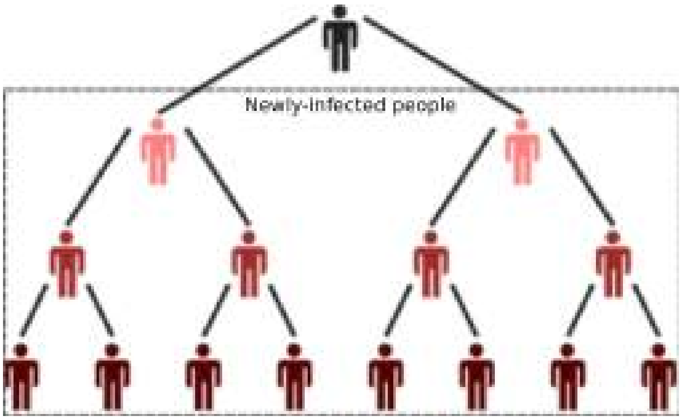
See video: 285K at www.everythingmaths.co.za

This means that the ratio between consecutive numbers in a geometric sequence is a constant (positive or negative). We will explain what we mean by ratio after looking at the following example.

Example: A flu epidemic EMCDS

Influenza (commonly called "flu") is caused by the influenza virus, which infects the respiratory tract (nose, throat, lungs). It can cause mild to severe illness that most of us get during winter time. The influenza virus is spread from person to person in respiratory droplets of coughs and sneezes. This is called "droplet spread". This can happen when droplets from a cough or sneeze of an infected person are propelled through the air and deposited on the mouth or nose of people nearby. It is good practice to cover your mouth when you cough or sneeze so as not to infect others around you when you have the flu. Regular hand washing is an effective way to prevent the spread of infection and illness.

Assume that you have the flu virus, and you forgot to cover your mouth when two friends came to visit while you were sick in bed. They leave, and the next day they also have the flu. Let's assume that each friend in turn spreads the virus to two of their friends by the same droplet spread the following day. Assuming this pattern continues and each sick person infects 2 other friends, we can represent these events in the following manner:



Each person infects two more people with the flu virus.

We can tabulate the events and formulate an equation for the general case:

Day (n)	**No. of newly-infected people**
1	$2 = 2$
2	$4 = 2 \times 2 = 2 \times 2^1$
3	$8 = 2 \times 4 = 2 \times 2 \times 2 = 2 \times 2^2$
4	$16 = 2 \times 8 = 2 \times 2 \times 2 \times 2 = 2 \times 2^3$
5	$32 = 2 \times 16 = 2 \times 2 \times 2 \times 2 \times 2 = 2 \times 2^4$
$\vdots$	$\vdots$
n	$2 \times 2 \times 2 \times 2 \times \cdots \times 2 = 2 \times 2^{n-1}$

The above table represents the number of **newly-infected** people after n days since you first infected your 2 friends.

You sneeze and the virus is carried over to 2 people who start the chain ($a = 2$). The next day, each one then infects 2 of their friends. Now 4 people are newly-infected. Each of them infects 2 people the third day, and 8 new people are infected, and so on. These events can be written as a geometric sequence:

$$2; 4; 8; 16; 32; \ldots$$

Note the constant ratio ($r = 2$) between the events. Recall from the linear arithmetic sequence how the common difference between terms was established. In the geometric sequence we can determine the constant ratio (r) from:

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} = r$$

More generally,

$$\frac{T_n}{T_{n-1}} = r$$

Exercise 1 – 4: Constant ratio of a geometric sequence

Determine the constant ratios for the following geometric sequences and write down the next three terms in each sequence:

1. $5; 10; 20; \ldots$
2. $\frac{1}{2}; \frac{1}{4}; \frac{1}{8}; \ldots$
3. $7; 0{,}7; 0{,}07; \ldots$
4. $p; 3p^2; 9p^3; \ldots$
5. $-3; 30; -300; \ldots$
6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 285M 2. 285N 3. 285P 4. 285Q 5. 285R

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The general term for a geometric sequence EMCDT

From the flu example above we know that $T_1 = 2$ and $r = 2$, and we have seen from the table that the n^{th} term is given by $T_n = 2 \times 2^{n-1}$.

The general geometric sequence can be expressed as:

$$\begin{aligned}
T_1 &= a & &= ar^0 \\
T_2 &= a \times r & &= ar^1 \\
T_3 &= a \times r \times r & &= ar^2 \\
T_4 &= a \times r \times r \times r & &= ar^3 \\
T_n &= a \times [r \times r \ldots (n-1) \text{ times}] & &= ar^{n-1}
\end{aligned}$$

Therefore the general formula for a geometric sequence is:

$$T_n = ar^{n-1}$$

where

- a is the first term in the sequence;
- r is the constant ratio.

Test for a geometric sequence

To test whether a sequence is a geometric sequence or not, check if the ratio between any two consecutive terms is constant:

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_n}{T_{n-1}} = r$$

If this condition does not hold, then the sequence is not a geometric sequence.

Exercise 1 – 5: General term of a geometric sequence

Determine the general formula for the n^{th} term of each of the following geometric sequences:

1. $5; 10; 20; \ldots$
2. $\frac{1}{2}; \frac{1}{4}; \frac{1}{8}; \ldots$
3. $7; 0{,}7; 0{,}07; \ldots$
4. $p; 3p^2; 9p^3; \ldots$
5. $-3; 30; -300; \ldots$
6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 285S 2. 285T 3. 285V 4. 285W 5. 285X

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Worked example 3: Flu epidemic

QUESTION

We continue with the previous flu example, where T_n is the number of newly-infected people after n days:

$$T_n = 2 \times 2^{n-1}$$

1. Calculate how many newly-infected people there are on the tenth day.
2. On which day will $16\ 384$ people be newly-infected?

SOLUTION

Step 1: Write down the known values and the general formula

$$\begin{aligned} a &= 2 \\ r &= 2 \\ T_n &= 2 \times 2^{n-1} \end{aligned}$$

Step 2: Use the general formula to calculate T_{10}

Substitute $n = 10$ into the general formula:

$$\begin{aligned} T_n &= a \times r^{n-1} \\ \therefore T_{10} &= 2 \times 2^{10-1} \\ &= 2 \times 2^9 \\ &= 2 \times 512 \\ &= 1024 \end{aligned}$$

On the tenth day, there are 1024 newly-infected people.

Step 3: Use the general formula to calculate n

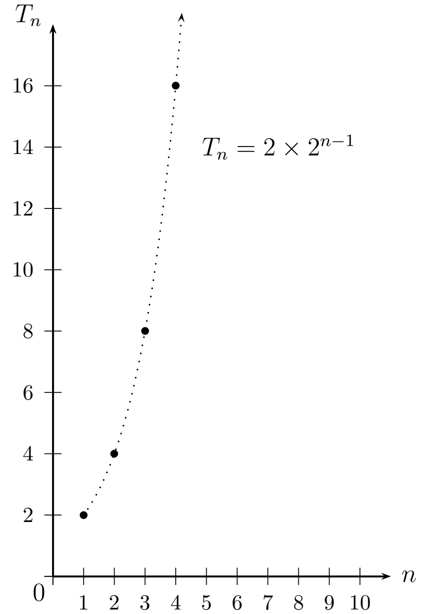
We know that $T_n = 16\ 384$ and can use the general formula to calculate the corresponding value of n:

$$\begin{aligned} T_n &= ar^{n-1} \\ 16\ 384 &= 2 \times 2^{n-1} \\ \frac{16\ 384}{2} &= 2^{n-1} \\ 8192 &= 2^{n-1} \\ \text{We can write } 8192 \text{ as } 2^{13} & \\ \text{So } 2^{13} &= 2^{n-1} \\ \therefore 13 &= n - 1 \quad \text{(same bases)} \\ \therefore n &= 14 \end{aligned}$$

There are $16\ 384$ newly-infected people on the 14^{th} day.

For this geometric sequence, plotting the number of newly-infected people (T_n) vs. the number of days (n) results in the following graph:

Day (n)	**No. of newly-infected people**
1	2
2	4
3	8
4	16
5	32
6	64
n	$2 \times 2^{n-1}$



In this example we are only dealing with positive integers ($n \in \{1; 2; 3; \ldots\}, T_n \in \{1; 2; 3; \ldots\}$), therefore the graph is not continuous and we do not join the points with a curve (the dotted line has been drawn to indicate the shape of an exponential graph).

Geometric mean

The geometric mean between two numbers is the value that forms a geometric sequence together with the two numbers.

For example, the geometric mean between 5 and 20 is the number that has to be inserted between 5 and 20 to form the geometric sequence: $5; x; 20$

$$\begin{aligned} \text{Determine the constant ratio: } \frac{x}{5} &= \frac{20}{x} \\ \therefore x^2 &= 20 \times 5 \\ x^2 &= 100 \\ x &= \pm 10 \end{aligned}$$

Important: remember to include both the positive and negative square root. The geometric mean generates two possible geometric sequences:

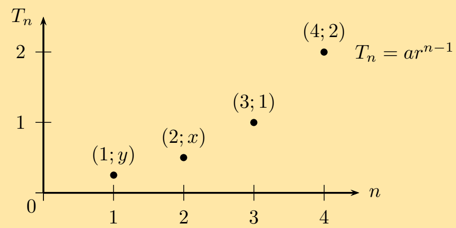
$$5; 10; 20; \ldots$$
$$5; -10; 20; \ldots$$

In general, the geometric mean (x) between two numbers a and b forms a geometric sequence with a and b:

$$\begin{aligned} \text{For a geometric sequence: } & a; x; b \\ \text{Determine the constant ratio: } \frac{x}{a} &= \frac{b}{x} \\ x^2 &= ab \\ \therefore x &= \pm\sqrt{ab} \end{aligned}$$

Exercise 1 – 6: Mixed exercises

1. The n^{th} term of a sequence is given by the formula $T_n = 6\left(\frac{1}{3}\right)^{n-1}$.
   a) Write down the first three terms of the sequence.
   b) What type of sequence is this?
2. Consider the following terms:
$$(k-4); (k+1); m; 5k$$
The first three terms form an arithmetic sequence and the last three terms form a geometric sequence. Determine the values of k and m if both are positive integers.
[IEB, Nov 2006]
3. Given a geometric sequence with second term $\frac{1}{2}$ and ninth term 64.
   a) Determine the value of r.
   b) Find the value of a.
   c) Determine the general formula of the sequence.
4. The diagram shows four sets of values of consecutive terms of a geometric sequence with the general formula $T_n = ar^{n-1}$.



   a) Determine a and r.
   b) Find x and y.
   c) Find the fifth term of the sequence.
5. Write down the next two terms for the following sequence:
$$1; \sin\theta; 1-\cos^2\theta; \ldots$$
6. $5; x; y$ is an arithmetic sequence and $x; y; 81$ is a geometric sequence. All terms in the sequences are integers. Calculate the values of x and y.
7. The two numbers $2x^2y^2$ and $8x^4$ are given.
   a) Write down the geometric mean between the two numbers in terms of x and y.
   b) Determine the constant ratio of the resulting sequence.
8. Insert three geometric means between -1 and $-\frac{1}{81}$. Give all possible answers.
9. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 285Y 2. 285Z 3. 2862 4. 2863 5. 2864 6. 2865
7. 2866 8. 2867

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1.3 Series EMCDV

It is often important and valuable to determine the sum of the terms of an arithmetic or geometric sequence. The sum of any sequence of numbers is called a series.

Finite series

We use the symbol S_n for the sum of the first n terms of a sequence $\{T_1; T_2; T_3; \ldots; T_n\}$:

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n$$

If we sum only a finite number of terms, we get a finite series.

For example, consider the following sequence of numbers

$$1; 4; 9; 16; 25; 36; 49; \ldots$$

We can calculate the sum of the first four terms:

$$S_4 = 1 + 4 + 9 + 16 = 30$$

This is an example of a finite series since we are only summing four terms.

Infinite series

If we sum infinitely many terms of a sequence, we get an infinite series:

$$S_\infty = T_1 + T_2 + T_3 + \cdots$$

Sigma notation EMCDW

Sigma notation is a very useful and compact notation for writing the sum of a given number of terms of a sequence.

A sum may be written out using the summation symbol $\sum$ (Sigma), which is the capital letter "S" in the Greek alphabet. It indicates that you must sum the expression to the right of the summation symbol:

For example,

$$\sum_{n=1}^{5} 2n = 2 + 4 + 6 + 8 + 10 = 30$$

In general,

$$\sum_{i=m}^{n} T_i = T_m + T_{m+1} + \cdots + T_{n-1} + T_n$$

where

- i is the index of the sum;
- m is the lower bound (or start index), shown below the summation symbol;
- n is the upper bound (or end index), shown above the summation symbol;
- T_i is a term of a sequence;
- the number of terms in the series $=$ end index $-$ start index $+ 1$.

The index i increases from m to n by steps of 1.

Note that this is also sometimes written as:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

When we write out all the terms in a sum, it is referred to as the expanded form.

If we are summing from $i = 1$ (which implies summing from the first term in a sequence), then we can use either S_n or $\sum$ notation:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \quad (n \text{ terms})$$

Worked example 4: Sigma notation

QUESTION

Expand the sequence and find the value of the series:

$$\sum_{n=1}^{6} 2^n$$

SOLUTION

Step 1: Expand the formula and write down the first six terms of the sequence

$$\begin{aligned}\sum_{n=1}^{6} 2^n &= 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 \quad (6 \text{ terms}) \\ &= 2 + 4 + 8 + 16 + 32 + 64\end{aligned}$$

This is a geometric sequence $2; 4; 8; 16; 32; 64$ with a constant ratio of 2 between consecutive terms.

Step 2: Determine the sum of the first six terms of the sequence

$$\begin{aligned}S_6 &= 2 + 4 + 8 + 16 + 32 + 64 \\ &= 126\end{aligned}$$

Worked example 5: Sigma notation

QUESTION

Find the value of the series:

$$\sum_{n=3}^{7} 2an$$

SOLUTION

Step 1: Expand the sequence and write down the five terms

$$\begin{aligned}\sum_{n=3}^{7} 2an &= 2a(3) + 2a(4) + 2a(5) + 2a(6) + 2a(7) \quad (5 \text{ terms})\\ &= 6a + 8a + 10a + 12a + 14a\end{aligned}$$

Step 2: Determine the sum of the five terms of the sequence

$$\begin{aligned}S_5 &= 6a + 8a + 10a + 12a + 14a\\ &= 50a\end{aligned}$$

Worked example 6: Sigma notation

QUESTION

Write the following series in sigma notation:

$$31 + 24 + 17 + 10 + 3$$

SOLUTION

Step 1: Consider the series and determine if it is an arithmetic or geometric series

First test for an arithmetic series: is there a common difference?

We let:

$$\begin{aligned}T_1 &= 31; & T_4 &= 10;\\ T_2 &= 24; & T_5 &= 3;\\ T_3 &= 17;\end{aligned}$$

We calculate:

$$\begin{aligned}d &= T_2 - T_1\\ &= 24 - 31\\ &= -7\\ d &= T_3 - T_2\\ &= 17 - 24\\ &= -7\end{aligned}$$

There is a common difference of -7, therefore this is an arithmetic series.

Step 2: Determine the general formula of the series

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 31 + (n-1)(-7) \\ &= 31 - 7n + 7 \\ &= -7n + 38 \end{aligned}$$

Be careful: brackets must be used when substituting $d = -7$ into the general term. Otherwise the equation would be $T_n = 31 + (n-1) - 7$, which would be incorrect.

Step 3: Determine the sum of the series and write in sigma notation

$$31 + 24 + 17 + 10 + 3 = 85$$
$$\therefore \sum_{n=1}^{5} (-7n + 38) = 85$$

Rules for sigma notation

1. Given two sequences, a_i and b_i:
$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$
2. For any constant c that is not dependent on the index i:
$$\begin{aligned} \sum_{i=1}^{n} (c \cdot a_i) &= c \cdot a_1 + c \cdot a_2 + c \cdot a_3 + \cdots + c \cdot a_n \\ &= c\,(a_1 + a_2 + a_3 + \cdots + a_n) \\ &= c \sum_{i=1}^{n} a_i \end{aligned}$$
3. Be accurate with the use of brackets:

   Example 1:
$$\begin{aligned} \sum_{n=1}^{3} (2n+1) &= 3 + 5 + 7 \\ &= 15 \end{aligned}$$

   Example 2:
$$\begin{aligned} \sum_{n=1}^{3} (2n) + 1 &= (2 + 4 + 6) + 1 \\ &= 13 \end{aligned}$$

   Note: the series in the second example has the general term $T_n = 2n$ and the $+1$ is added to the sum of the three terms. It is very important in sigma notation to use brackets correctly.

4. $$\sum_{i=m}^{n} a_i$$

The values of i:

- start at m (m is not always 1);
- increase in steps of 1;
- and end at n.

Exercise 1 – 7: Sigma notation

1. Determine the value of the following:

   a) $$\sum_{k=1}^{4} 2$$

   b) $$\sum_{i=-1}^{3} i$$

   c) $$\sum_{n=2}^{5} (3n-2)$$

2. Expand the series:

   a) $$\sum_{k=1}^{6} 0^k$$

   b) $$\sum_{n=-3}^{0} 8$$

   c) $$\sum_{k=1}^{5} (ak)$$

3. Calculate the value of a:

   a) $$\sum_{k=1}^{3} \left(a \,.\, 2^{k-1}\right) = 28$$

   b) $$\sum_{j=1}^{4} \left(2^{-j}\right) = a$$

4. Write the following in sigma notation:
$$\frac{1}{9} + \frac{1}{3} + 1 + 3$$

5. Write the sum of the first 25 terms of the series below in sigma notation:
$$11 + 4 - 3 - 10 \ldots$$

6. Write the sum of the first 1000 natural, odd numbers in sigma notation.

7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 2868 1b. 2869 1c. 286B 2a. 286C 2b. 286D 2c. 286F
3a. 286G 3b. 286H 4. 286J 5. 286K 6. 286M

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1.4 Finite arithmetic series EMCDX

An arithmetic sequence is a sequence of numbers, such that the difference between any term and the previous term is a constant number called the common difference (d):

$$T_n = a + (n-1)\,d$$

where

- T_n is the n^{th} term of the sequence;
- a is the first term;
- d is the common difference.

When we sum a finite number of terms in an arithmetic sequence, we get a finite arithmetic series.

The sum of the first one hundred integers

A simple arithmetic sequence is when $a = 1$ and $d = 1$, which is the sequence of positive integers:

$$\begin{aligned} T_n &= a + (n-1)\,d \\ &= 1 + (n-1)\,(1) \\ &= n \\ \therefore \{T_n\} &= 1; 2; 3; 4; 5; \ldots \end{aligned}$$

If we wish to sum this sequence from $n = 1$ to any positive integer, for example 100, we would write

$$\sum_{n=1}^{100} n = 1 + 2 + 3 + \cdots + 100$$

This gives the answer to the sum of the first 100 positive integers.

The mathematician, Karl Friedrich Gauss, discovered the following proof when he was only 8 years old. His teacher had decided to give his class a problem which would distract them for the entire day by asking them to add all the numbers from 1 to 100. Young Karl quickly realised how to do this and shocked the teacher with the correct answer, 5050. This is the method that he used:

- Write the numbers in ascending order.
- Write the numbers in descending order.
- Add the corresponding pairs of terms together.
- Simplify the equation by making S_n the subject of the equation.

$$
\begin{aligned}
S_{100} &= 1 \quad + \quad 2 \quad + \quad 3 + \cdots + 98 + 99 + 100 \\
+ \quad \underline{S_{100}} &= \underline{100 + 99 + 98 + \cdots + 3 \quad + \quad 2 \quad + \quad 1} \\
\therefore 2S_{100} &= 101 + 101 + 101 + \cdots + 101 + 101 + 101 \\
\therefore 2S_{100} &= 101 \times 100 \\
&= 10\ 100 \\
\therefore S_{100} &= \frac{10\ 100}{2} \\
&= 5050
\end{aligned}
$$

General formula for a finite arithmetic series EMCDY

If we sum an arithmetic sequence, it takes a long time to work it out term-by-term. We therefore derive the general formula for evaluating a finite arithmetic series. We start with the general formula for an arithmetic sequence of n terms and sum it from the first term (a) to the last term in the sequence (l):

$$
\begin{aligned}
\sum_{n=1}^{l} T_n &= S_n \\
S_n &= a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l \\
+ \quad \underline{S_n} &= \underline{l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a} \\
\therefore 2S_n &= (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) + (a+l) \\
\therefore 2S_n &= n \times (a+l) \\
\therefore S_n &= \frac{n}{2}(a+l)
\end{aligned}
$$

This general formula is useful if the last term in the series is known.

We substitute $l = a + (n-1)d$ into the above formula and simplify:

$$
\begin{aligned}
S_n &= \frac{n}{2}(a + [a + (n-1)d]) \\
\therefore S_n &= \frac{n}{2}[2a + (n-1)d]
\end{aligned}
$$

The general formula for determining the sum of an arithmetic series is given by:

$$S_n = \frac{n}{2}\left[2a + (n-1)\,d\right]$$

or

$$S_n = \frac{n}{2}(a+l)$$

For example, we can calculate the sum S_{20} for the arithmetic sequence $T_n = 3 + 7\,(n-1)$ by summing all the individual terms:

$$
\begin{aligned}
S_{20} &= \sum_{n=1}^{20} [3 + 7\,(n-1)] \\
&= 3 + 10 + 17 + 24 + 31 + 38 + 45 + 52 \\
&+ 59 + 66 + 73 + 80 + 87 + 94 + 101 \\
&+ 108 + 115 + 122 + 129 + 136 \\
&= 1390
\end{aligned}
$$

or, more sensibly, we could use the general formula for determining an arithmetic series by substituting $a = 3$, $d = 7$ and $n = 20$:

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ S_{20} &= \frac{20}{2}\left[2(3) + 7\left(20 - 1\right)\right] \\ &= 1390 \end{aligned}$$

This example demonstrates how useful the general formula for determining an arithmetic series is, especially when the series has a large number of terms.

See video: 286N at www.everythingmaths.co.za

Worked example 7: General formula for the sum of an arithmetic sequence

QUESTION

Find the sum of the first 30 terms of an arithmetic series with $T_n = 7n - 5$ by using the formula.

SOLUTION

Step 1: Use the general formula to generate terms of the sequence and write down the known variables

$$\begin{aligned} T_n &= 7n - 5 \\ \therefore T_1 &= 7(1) - 5 \\ &= 2 \\ T_2 &= 7(2) - 5 \\ &= 9 \\ T_3 &= 7(3) - 5 \\ &= 16 \end{aligned}$$

This gives the sequence: $2; 9; 16 \ldots$

$$a = 2; \quad d = 7; \quad n = 30$$

Step 2: Write down the general formula and substitute the known values

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ S_{30} &= \frac{30}{2}[2(2) + (30-1)(7)] \\ &= 15(4 + 203) \\ &= 15(207) \\ &= 3105 \end{aligned}$$

Step 3: Write the final answer

$S_{30} = 3105$

Worked example 8: Sum of an arithmetic sequence if first and last terms are known

QUESTION

Find the sum of the series $-5 - 3 - 1 + \cdots\cdots + 123$

SOLUTION

Step 1: Identify the type of series and write down the known variables

$$\begin{aligned} d &= T_2 - T_1 \\ &= -3 - (-5) \\ &= 2 \\ d &= T_3 - T_2 \\ &= -1 - (-3) \\ &= 2 \end{aligned}$$

$$a = -5; \quad d = 2; \quad l = 123$$

Step 2: Determine the value of n

$$\begin{aligned} T_n &= a + (n-1)d \\ \therefore 123 &= -5 + (n-1)(2) \\ &= -5 + 2n - 2 \\ \therefore 130 &= 2n \\ \therefore n &= 65 \end{aligned}$$

Step 3: Use the general formula to find the sum of the series

$$\begin{aligned} S_n &= \frac{n}{2}(a + l) \\ S_{65} &= \frac{65}{2}(-5 + 123) \\ &= \frac{65}{2}(118) \\ &= 3835 \end{aligned}$$

Step 4: Write the final answer

$S_{65} = 3835$

Worked example 9: Finding n given the sum of an arithmetic sequence

QUESTION

Given an arithmetic sequence with $T_2 = 7$ and $d = 3$, determine how many terms must be added together to give a sum of 2146.

SOLUTION

Step 1: Write down the known variables

$$\begin{aligned} d &= T_2 - T_1 \\ \therefore 3 &= 7 - a \\ \therefore a &= 4 \end{aligned}$$

$$a = 4; \quad d = 3; \quad S_n = 2146$$

Step 2: Use the general formula to determine the value of n

$$\begin{aligned}
S_n &= \frac{n}{2}(2a + (n-1)d) \\
2146 &= \frac{n}{2}(2(4) + (n-1)(3)) \\
4292 &= n(8 + 3n - 3) \\
\therefore 0 &= 3n^2 + 5n - 4292 \\
&= (3n + 116)(n - 37) \\
\therefore n = -\frac{116}{3} &\text{ or } n = 37
\end{aligned}$$

but n must be a positive integer, therefore $n = 37$.

We could have solved for n using the quadratic formula but factorising by inspection is usually the quickest method.

Step 3: Write the final answer

$S_{37} = 2146$

Worked example 10: Finding n given the sum of an arithmetic sequence

QUESTION

The sum of the second and third terms of an arithmetic sequence is equal to zero and the sum of the first 36 terms of the series is equal to 1152. Find the first three terms in the series.

SOLUTION

Step 1: Write down the given information

$$\begin{aligned}
T_2 + T_3 &= 0 \\
\text{So} \quad (a + d) + (a + 2d) &= 0 \\
\therefore 2a + 3d &= 0 \ldots\ldots (1)
\end{aligned}$$

$$\begin{aligned}
S_n &= \frac{n}{2}(2a + (n-1)d) \\
S_{36} &= \frac{36}{2}(2a + (36-1)d) \\
1152 &= 18(2a + 35d) \\
\therefore 64 &= 2a + 35d \ldots\ldots (2)
\end{aligned}$$

Step 2: Solve the two equations simultaneously

$$\begin{aligned}
2a + 3d &= 0 \ldots\ldots (1) \\
2a + 35d &= 64 \ldots\ldots (2) \\
\text{Eqn } (2) - (1): \quad 32d &= 64 \\
\therefore d &= 2 \\
\text{And } 2a + 3(2) &= 0 \\
2a &= -6 \\
\therefore a &= -3
\end{aligned}$$

Step 3: Write the final answer

The first three terms of the series are:

$$\begin{aligned} T_1 &= a = -3 \\ T_2 &= a + d = -3 + 2 = -1 \\ T_3 &= a + 2d = -3 + 2(2) = 1 \end{aligned}$$

$$-3 - 1 + 1$$

Calculating the value of a term given the sum of n terms:

If the first term in a series is T_1, then $S_1 = T_1$.

We also know the sum of the first two terms $S_2 = T_1 + T_2$, which we rearrange to make T_2 the subject of the equation:

$$\begin{aligned} T_2 &= S_2 - T_1 \\ \text{Substitute } S_1 &= T_1 \\ \therefore T_2 &= S_2 - S_1 \end{aligned}$$

Similarly, we could determine the third and fourth term in a series:

$$\begin{aligned} T_3 &= S_3 - S_2 \\ \text{And } T_4 &= S_4 - S_3 \end{aligned}$$

$T_n = S_n - S_{n-1},$ for $n \in \{2; 3; 4; \ldots\}$ and $T_1 = S_1$

Exercise 1 – 8: Sum of an arithmetic series

1. Determine the value of k:

$$\sum_{n=1}^{k} (-2n) = -20$$

2. The sum to n terms of an arithmetic series is $S_n = \frac{n}{2}(7n + 15)$.

    a) How many terms of the series must be added to give a sum of 425?

    b) Determine the sixth term of the series.

3. a) The common difference of an arithmetic series is 3. Calculate the values of n for which the n^{th} term of the series is 93, and the sum of the first n terms is 975.

    b) Explain why there are two possible answers.

4. The third term of an arithmetic sequence is -7 and the seventh term is 9. Determine the sum of the first 51 terms of the sequence.

5. Calculate the sum of the arithmetic series $4 + 7 + 10 + \cdots + 901$.

6. Evaluate without using a calculator: $\dfrac{4 + 8 + 12 + \cdots + 100}{3 + 10 + 17 + \cdots + 101}$

7. The second term of an arithmetic sequence is -4 and the sum of the first six terms of the series is 21.
   a) Find the first term and the common difference.
   b) Hence determine T_{100}.
   [IEB, Nov 2004]
8. Determine the value of the following:
   a) $$\sum_{w=0}^{8}(7w+8)$$
   b) $$\sum_{j=1}^{8}7j+8$$
9. Determine the value of n.
$$\sum_{c=1}^{n}(2-3c)=-330$$
10. The sum of n terms of an arithmetic series is $5n^2-11n$ for all values of n. Determine the common difference.
11. The sum of an arithmetic series is 100 times its first term, while the last term is 9 times the first term. Calculate the number of terms in the series if the first term is not equal to zero.
12. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 286P 2a. 286Q 2b. 286R 3. 286S 4. 286T 5. 286V
6. 286W 7. 286X 8a. 286Y 8b. 286Z 9. 2872 10. 2873
11. 2874

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1.5 Finite geometric series EMCDZ

When we sum a known number of terms in a geometric sequence, we get a finite geometric series. We generate a geometric sequence using the general form:

$$T_n = a\ .\ r^{n-1}$$

where

- n is the position of the sequence;
- T_n is the n^{th} term of the sequence;
- a is the first term;
- r is the constant ratio.

General formula for a finite geometric series EMCF2

$$
\begin{aligned}
S_n &= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \ldots (1) \\
r \times S_n &= \qquad ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \ldots\ldots (2)
\end{aligned}
$$

Subtract eqn. (2) from eqn. (1)

$$
\begin{aligned}
\therefore S_n - rS_n &= a + 0 + 0 + \cdots - ar^n \\
S_n - rS_n &= a - ar^n \\
S_n(1-r) &= a(1-r^n) \\
\therefore S_n &= \frac{a(1-r^n)}{1-r} \quad \text{(where } r \neq 1\text{)}
\end{aligned}
$$

The general formula for determining the sum of a geometric series is given by:

$$
S_n = \frac{a(1-r^n)}{1-r} \qquad \text{where } r \neq 1
$$

This formula is easier to use when $r < 1$.

Alternative formula:

$$
\begin{aligned}
S_n &= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \ldots\ldots (1) \\
r \times S_n &= \qquad ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \ldots\ldots (2)
\end{aligned}
$$

Subtract eqn. (1) from eqn. (2)

$$
\begin{aligned}
\therefore rS_n - S_n &= ar^n - a \\
S_n(r-1) &= a(r^n - 1) \\
\therefore S_n &= \frac{a(r^n-1)}{r-1} \quad \text{(where } r \neq 1\text{)}
\end{aligned}
$$

The general formula for determining the sum of a geometric series is given by:

$$
S_n = \frac{a(r^n-1)}{r-1} \qquad \text{where } r \neq 1
$$

This formula is easier to use when $r > 1$.

Worked example 11: Sum of a geometric series

QUESTION

Calculate:

$$\sum_{k=1}^{6} 32\left(\frac{1}{2}\right)^{k-1}$$

SOLUTION

Step 1: Write down the first three terms of the series

$$\begin{aligned} k = 1; \quad T_1 &= 32\left(\frac{1}{2}\right)^{0} = 32 \\ k = 2; \quad T_2 &= 32\left(\frac{1}{2}\right)^{2-1} = 16 \\ k = 3; \quad T_3 &= 32\left(\frac{1}{2}\right)^{3-1} = 8 \end{aligned}$$

We have generated the series $32 + 16 + 8 + \cdots$

Step 2: Determine the values of a and r

$$\begin{aligned} a &= T_1 = 32 \\ r &= \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{1}{2} \end{aligned}$$

Step 3: Use the general formula to find the sum of the series

$$\begin{aligned} S_n &= \frac{a(1 - r^n)}{1 - r} \\ S_6 &= \frac{32(1 - \left(\frac{1}{2}\right)^6)}{1 - \frac{1}{2}} \\ &= \frac{32\left(1 - \frac{1}{64}\right)}{\frac{1}{2}} \\ &= 2 \times 32\left(\frac{63}{64}\right) \\ &= 64\left(\frac{63}{64}\right) \\ &= 63 \end{aligned}$$

Step 4: Write the final answer

$$\sum_{k=1}^{6} 32\left(\frac{1}{2}\right)^{k-1} = 63$$

See video: 2875 at www.everythingmaths.co.za

Worked example 12: Sum of a geometric series

QUESTION

Given a geometric series with $T_1 = -4$ and $T_4 = 32$. Determine the values of r and n if $S_n = 84$.

SOLUTION

Step 1: Determine the values of a and r

$$\begin{aligned} a = T_1 &= -4 \\ T_4 = ar^3 &= 32 \\ \therefore -4r^3 &= 32 \\ r^3 &= -8 \\ \therefore r &= -2 \end{aligned}$$

Therefore the geometric series is $-4 + 8 - 16 + 32 \ldots$ Notice that the signs of the terms alternate because $r < 0$.

We write the general term for this series as $T_n = -4(-2)^{n-1}$.

Step 2: Use the general formula for the sum of a geometric series to determine the value of n

$$\begin{aligned} S_n &= \frac{a(1 - r^n)}{1 - r} \\ \therefore 84 &= \frac{-4(1 - (-2)^n)}{1 - (-2)} \\ 84 &= \frac{-4(1 - (-2)^n)}{3} \\ -\frac{3}{4} \times 84 &= 1 - (-2)^n \\ -63 &= 1 - (-2)^n \\ (-2)^n &= 64 \\ (-2)^n &= (-2)^6 \\ \therefore n &= 6 \end{aligned}$$

Step 3: Write the final answer

$$r = -2 \text{ and } n = 6$$

Worked example 13: Sum of a geometric series

QUESTION

Use the general formula for the sum of a geometric series to determine k if

$$\sum_{n=1}^{8} k\left(\frac{1}{2}\right)^{n} = \frac{255}{64}$$

SOLUTION

Step 1: Write down the first three terms of the series

$$\begin{aligned} n = 1; \quad T_1 &= k\left(\frac{1}{2}\right)^{1} = \frac{1}{2}k \\ n = 2; \quad T_2 &= k\left(\frac{1}{2}\right)^{2} = \frac{1}{4}k \\ n = 3; \quad T_3 &= k\left(\frac{1}{2}\right)^{3} = \frac{1}{8}k \end{aligned}$$

We have generated the series $\frac{1}{2}k + \frac{1}{4}k + \frac{1}{8}k + \cdots$

We can take out the common factor k and write the series as: $k\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)$

$$\therefore k\sum_{n=1}^{8} \left(\frac{1}{2}\right)^{n} = \frac{255}{64}$$

Step 2: Determine the values of a and r

$$\begin{aligned} a &= T_1 = \frac{1}{2} \\ r &= \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{1}{2} \end{aligned}$$

Step 3: Calculate the sum of the first eight terms of the geometric series

$$\begin{aligned} \therefore S_n &= \frac{a(1 - r^n)}{1 - r} \\ S_8 &= \frac{\frac{1}{2}(1 - \left(\frac{1}{2}\right)^8)}{1 - \frac{1}{2}} \\ &= \frac{\frac{1}{2}(1 - \left(\frac{1}{2}\right)^8)}{\frac{1}{2}} \\ &= 1 - \frac{1}{256} \\ &= \frac{255}{256} \end{aligned}$$

$$\therefore \sum_{n=1}^{8} \left(\frac{1}{2}\right)^{n} = \frac{255}{256}$$

So then we can write:

$$
\begin{aligned}
k\sum_{n=1}^{8}\left(\frac{1}{2}\right)^{n} &= \frac{255}{64}\\
k\left(\frac{255}{256}\right) &= \frac{255}{64}\\
\therefore k &= \frac{255}{64}\times\frac{256}{255}\\
&= \frac{256}{64}\\
&= 4
\end{aligned}
$$

Step 4: Write the final answer

$$k = 4$$

Exercise 1 – 9: Sum of a geometric series

1. Prove that $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$ and state any restrictions.
2. Given the geometric sequence $1; -3; 9; \ldots$ determine:
   a) The eighth term of the sequence.
   b) The sum of the first eight terms of the sequence.
3. Determine:
$$\sum_{n=1}^{4} 3 \,.\, 2^{n-1}$$
4. Find the sum of the first 11 terms of the geometric series $6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$
5. Show that the sum of the first n terms of the geometric series $54 + 18 + 6 + \cdots + 5\left(\frac{1}{3}\right)^{n-1}$ is given by $\left(81 - 3^{4-n}\right)$.
6. The eighth term of a geometric sequence is 640. The third term is 20. Find the sum of the first 7 terms.
7. Given:
$$\sum_{t=1}^{n} 8\left(\frac{1}{2}\right)^{t}$$
   a) Find the first three terms in the series.
   b) Calculate the number of terms in the series if $S_n = 7\frac{63}{64}$.
8. The ratio between the sum of the first three terms of a geometric series and the sum of the 4^{th}, 5^{th} and 6^{th} terms of the same series is $8 : 27$. Determine the constant ratio and the first 2 terms if the third term is 8.
9. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 2876 2a. 2877 2b. 2878 3. 2879 4. 287B 5. 287C
6. 287D 7a. 287F 7b. 287G 8. 287H

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So far we have been working only with finite sums, meaning that whenever we determined the sum of a series, we only considered the sum of the first n terms. We now consider what happens when we add an infinite number of terms together. Surely if we sum infinitely many numbers, no matter how small they are, the answer goes to infinity? In some cases the answer does indeed go to infinity (like when we sum all the positive integers), but surprisingly there are some cases where the answer is a finite real number.

Investigation: Sum of an infinite series

1. Cut a piece of string 1 m in length.
2. Now cut the piece of string in half and place one half on the desk.
3. Cut the other half in half again and put one of the pieces on the desk.
4. Repeat this process until the piece of string is too short to cut easily.
5. Draw a diagram to illustrate the sequence of lengths of the pieces of string.
6. Can this sequence be expressed mathematically? Hint: express the shorter lengths of string as a fraction of the original length of string.
7. What is the sum of the lengths of all the pieces of string?
8. Predict what would happen if these steps could be repeated infinitely many times.
9. Will the sum of the lengths of string ever be greater than 1?
10. What can you conclude?

Worked example 14: Sum to infinity

QUESTION

Complete the table below for the geometric series $T_n = \left(\frac{1}{2}\right)^n$ and answer the questions that follow:

	Terms	S_n	$1 - S_n$
T_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$T_1 + T_2$			
$T_1 + T_2 + T_3$			
$T_1 + T_2 + T_3 + T_4$			

1. As more and more terms are added, what happens to the value of S_n?
2. As more more and more terms are added, what happens to the value of $1 - S_n$?
3. Predict the maximum value of S_n for the sum of infinitely many terms in the series.

SOLUTION

Step 1: Complete the table

	Terms	S_n	$1 - S_n$
T_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$T_1 + T_2$	$\frac{1}{2} + \frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$
$T_1 + T_2 + T_3$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$\frac{7}{8}$	$\frac{1}{8}$
$T_1 + T_2 + T_3 + T_4$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$	$\frac{15}{16}$	$\frac{1}{16}$

Step 2: Consider the value of S_n and $1 - S_n$

As more terms in the series are added together, the value of S_n increases:

$$\frac{1}{2} \quad < \quad \frac{3}{4} \quad < \quad \frac{7}{8} \quad < \quad \cdots$$

However, by considering $1 - S_n$, we notice that the amount by which S_n increases gets smaller and smaller as more terms are added:

$$\frac{1}{2} \quad > \quad \frac{1}{4} \quad > \quad \frac{1}{8} \quad > \quad \cdots$$

We can therefore conclude that the value of S_n is approaching a maximum value of 1; it is converging to 1.

Step 3: Write conclusion mathematically

We can conclude that the sum of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

gets closer to 1 ($S_n \to 1$) as the number of terms approaches infinity ($n \to \infty$), therefore the series converges.

$$\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = 1$$

We express the sum of an infinite number of terms of a series as

$$S_\infty = \sum_{i=1}^{\infty} T_i$$

Convergence and divergence

If the sum of a series gets closer and closer to a certain value as we increase the number of terms in the sum, we say that the series converges. In other words, there is a limit to the sum of a converging series. If a series does not converge, we say that it diverges. The sum of an infinite series usually tends to infinity, but there are some special cases where it does not.

Exercise 1 – 10: Convergent and divergent series

For each of the general terms below:

- Determine if it forms an arithmetic or geometric series.
- Calculate S_1, S_2, S_{10} and S_{100}.
- Determine if the series is convergent or divergent.

1. $T_n = 2n$
2. $T_n = (-n)$
3. $T_n = (\frac{2}{3})^n$
4. $T_n = 2^n$
5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 287J 2. 287K 3. 287M 4. 287N

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Note the following:

- An arithmetic series never converges: as n tends to infinity, the series will always tend to positive or negative infinity.
- Some geometric series converge (have a limit) and some diverge (as n tends to infinity, the series does not tend to any limit or it tends to infinity).

Infinite geometric series EMCF4

There is a simple test for determining whether a geometric series converges or diverges; if $-1 < r < 1$, then the infinite series will converge. If r lies outside this interval, then the infinite series will diverge.

Test for convergence:

- If $-1 < r < 1$, then the infinite geometric series converges.
- If $r < -1$ or $r > 1$, then the infinite geometric series diverges.

We derive the formula for calculating the value to which a geometric series converges as follows:

$$S_n = \sum_{i=1}^{n} ar^{i-1} = \frac{a\left(1 - r^n\right)}{1 - r}$$

Now consider the behaviour of r^n for $-1 < r < 1$ as n becomes larger.

Let $r = \frac{1}{2}$:

$$\begin{aligned} n = 1 : r^n = r^1 &= \left(\frac{1}{2}\right)^1 = \frac{1}{2} \\ n = 2 : r^n = r^2 &= \left(\frac{1}{2}\right)^2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} < \frac{1}{2} \\ n = 3 : r^n = r^3 &= \left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} < \frac{1}{4} \end{aligned}$$

Since r is in the range $-1 < r < 1$, we see that r^n gets closer to 0 as n gets larger. Therefore $(1 - r^n)$ gets closer to 1.

Therefore,

$$\begin{aligned} S_n &= \frac{a\left(1 - r^n\right)}{1 - r} \\ \text{If } \quad -1 < r < 1, &\quad \text{then } r^n \to 0 \text{ as } n \to \infty \\ \therefore S_\infty &= \frac{a\left(1 - 0\right)}{1 - r} \\ &= \frac{a}{1 - r} \end{aligned}$$

The sum of an infinite geometric series is given by the formula

$$\therefore S_\infty = \sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1 - r} \qquad (-1 < r < 1)$$

where

- a is the first term of the series;
- r is the constant ratio.

Alternative notation:

$$\underbrace{S_n}_{n \to \infty} \to \frac{a}{1 - r} \quad \text{if } -1 < r < 1$$

In words: as the number of terms (n) tends to infinity, the sum of a converging geometric series (S_n) tends to the value $\frac{a}{1-r}$.

See video: 287P at www.everythingmaths.co.za

Worked example 15: Sum to infinity of a geometric series

QUESTION

Given the series $18 + 6 + 2 + \cdots$. Find the sum to infinity if it exists.

SOLUTION

Step 1: Determine the value of r

We need to know the value of r to determine whether the series converges or diverges.

$$\begin{aligned} \frac{T_2}{T_1} &= \frac{6}{18} \\ &= \frac{1}{3} \\ \frac{T_3}{T_2} &= \frac{2}{6} \\ &= \frac{1}{3} \\ \therefore r &= \frac{1}{3} \end{aligned}$$

Since $-1 < r < 1$, we can conclude that this is a convergent geometric series.

Step 2: Determine the sum to infinity

Write down the formula for the sum to infinity and substitute the known values:

$$a = 18; \qquad r = \frac{1}{3}$$

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{18}{1-\frac{1}{3}} \\ &= \frac{18}{\frac{2}{3}} \\ &= 18 \times \frac{3}{2} \\ &= 27 \end{aligned}$$

As n tends to infinity, the sum of this series tends to 27; no matter how many terms are added together, the value of the sum will never be greater than 27.

Worked example 16: Using the sum to infinity to convert recurring decimals to fractions

QUESTION

Use two different methods to convert the recurring decimal $0,\dot{5}$ to a proper fraction.

SOLUTION

Step 1: Convert the recurring decimal to a fraction using equations

$$\begin{aligned} \text{Let } x &= 0{,}\dot{5} \\ \therefore x &= 0{,}555\ldots\ldots(1) \\ 10x &= 5{,}55\ldots\ldots(2) \\ (2)-(1): \quad 9x &= 5 \\ \therefore x &= \frac{5}{9} \end{aligned}$$

Step 2: Convert the recurring decimal to a fraction using the sum to infinity

$$\begin{aligned} & 0{,}\dot{5} = 0{,}5 + 0{,}05 + 0{,}005 + \ldots \\ \text{or} \quad & 0{,}\dot{5} = \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \ldots \end{aligned}$$

This is a geometric series with $r = 0{,}1 = \frac{1}{10}$. And since $-1 < r < 1$, we can conclude that the series is convergent.

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{\frac{5}{10}}{1-\frac{1}{10}} \\ &= \frac{\frac{5}{10}}{\frac{9}{10}} \\ &= \frac{5}{9} \end{aligned}$$

Worked example 17: Sum to infinity

QUESTION

Determine the possible values of a and r if

$$\sum_{n=1}^{\infty} ar^{n-1} = 5$$

SOLUTION

Step 1: Write down the sum to infinity formula and substitute known values

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ \therefore 5 &= \frac{a}{1-r} \\ a &= 5(1-r) \\ \therefore a &= 5 - 5r \\ \text{And } 5r &= 5 - a \\ \therefore r &= \frac{5-a}{5} \end{aligned}$$

Step 2: Apply the condition for convergence to determine possible values of a

For a series to converge: $-1 < r < 1$

$$\begin{aligned} -1 < r &< 1 \\ -1 < \frac{5-a}{5} &< 1 \\ -5 < 5 - a &< 5 \\ -10 < -a &< 0 \\ 0 < a &< 10 \end{aligned}$$

Step 3: Write the final answer

For the series to converge, $0 < a < 10$ and $-1 < r < 1$.

Exercise 1 – 11:

1. What value does $\left(\frac{2}{5}\right)^n$ approach as n tends towards ∞?
2. Find the sum to infinity of the geometric series $3+1+\frac{1}{3}+\frac{1}{9}+\cdots$
3. Determine for which values of x, the geometric series $2+\frac{2}{3}\left(x+1\right)+\frac{2}{9}(x+1)^2+\cdots$ will converge.
4. The sum to infinity of a geometric series with positive terms is $4\frac{1}{6}$ and the sum of the first two terms is $2\frac{2}{3}$. Find a, the first term, and r, the constant ratio between consecutive terms.
5. Use the sum to infinity to show that $0{,}\dot{9}=1$.
6. A shrub 110 cm high is planted in a garden. At the end of the first year, the shrub is 120 cm tall. Thereafter the growth of the shrub each year is half of it's growth in the previous year. Show that the height of the shrub will never exceed 130 cm. Draw a graph of the relationship between time and growth.

   [IEB, Nov 2003]
7. Find p:

$$\sum_{k=1}^{\infty} 27p^k = \sum_{t=1}^{12}(24-3t)$$

8. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 287Q 2. 287R 3. 287S 4. 287T 5. 287V 6. 287W
7. 287X

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1.7 Summary EMCF5

Arithmetic sequence

- common difference (d) between any two consecutive terms: $d = T_n - T_{n-1}$
- general form: $a + (a + d) + (a + 2d) + \cdots$
- general formula: $T_n = a + (n - 1)d$
- graph of the sequence lies on a straight line

Quadratic sequence

- common second difference between any two consecutive terms
- general formula: $T_n = an^2 + bn + c$
- graph of the sequence lies on a parabola

Geometric sequence

- constant ratio (r) between any two consecutive terms: $r = \frac{T_n}{T_{n-1}}$
- general form: $a + ar + ar^2 + \cdots$
- general formula: $T_n = ar^{n-1}$
- graph of the sequence lies on an exponential curve

Sigma notation

$$\sum_{k=1}^{n} T_k$$

Sigma notation is used to indicate the sum of the terms given by T_k, starting from $k = 1$ and ending at $k = n$.

Series

- the sum of certain numbers of terms in a sequence
- arithmetic series:
  - $S_n = \frac{n}{2}[a + l]$
  - $S_n = \frac{n}{2}[2a + (n - 1)d]$
- geometric series:
  - $S_n = \frac{a(1-r^n)}{1-r}$ if $r < 1$
  - $S_n = \frac{a(r^n-1)}{r-1}$ if $r > 1$

Sum to infinity

A convergent geometric series, with $-1 < r < 1$, tends to a certain fixed number as the number of terms in the sum tends to infinity.

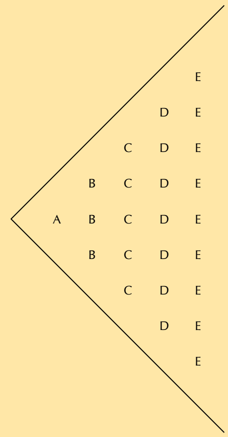
$$S_\infty = \sum_{n=1}^{\infty} T_n = \frac{a}{1 - r}$$

Exercise 1 – 12: End of chapter exercises

1. Is $1+2+3+4+\cdots$ an example of a *finite series* or an *infinite series*?
2. A new soccer competition requires each of 8 teams to play every other team once.
   a) Calculate the total number of matches to be played in the competition.
   b) If each of n teams played each other once, determine a formula for the total number of matches in terms of n.
3. Calculate:
$$\sum_{k=2}^{6} 3\left(\frac{1}{3}\right)^{k+2}$$
4. The first three terms of a convergent geometric series are: $x+1$; $x-1$; $2x-5$.
   a) Calculate the value of x, ($x \neq 1$ or 1).
   b) Sum to infinity of the series.
5. Write the sum of the first twenty terms of the following series in $\sum$ notation.
$$6+3+\frac{3}{2}+\frac{3}{4}+\cdots$$
6. Determine:
$$\sum_{k=1}^{\infty} 12\left(\frac{1}{5}\right)^{k-1}$$
7. A man was injured in an accident at work. He receives a disability grant of R 4800 in the first year. This grant increases with a fixed amount each year.
   a) What is the annual increase if he received a total of R $143\,500$ over 20 years?
   b) His initial annual expenditure is R 2600, which increases at a rate of R 400 per year. After how many years will his expenses exceed his income?
8. The length of the side of a square is 4 units. This square is divided into 4 equal, smaller squares. One of the smaller squares is then divided into four equal, even smaller squares. One of the even smaller squares is divided into four, equal squares. This process is repeated indefinitely. Calculate the sum of the areas of all the squares.
9. Thembi worked part-time to buy a Mathematics book which costs R $29{,}50$. On 1 February she saved R $1{,}60$, and every day saves 30 cents more than she saved the previous day. So, on the second day, she saved R $1{,}90$, and so on. After how many days did she have enough money to buy the book?
10. A plant reaches a height of 118 mm after one year under ideal conditions in a greenhouse. During the next year, the height increases by 12 mm. In each successive year, the height increases by $\frac{5}{8}$ of the previous year's growth. Show that the plant will never reach a height of more than 150 mm.
11. Calculate the value of n if:
$$\sum_{a=1}^{n} (20-4a) = -20$$

12. Michael saved R 400 during the first month of his working life. In each subsequent month, he saved 10% more than what he had saved in the previous month.
    a) How much did he save in the seventh working month?
    b) How much did he save all together in his first 12 working months?
13. The Cape Town High School wants to build a school hall and is busy with fundraising. Mr. Manuel, an ex-learner of the school and a successful politician, offers to donate money to the school. Having enjoyed mathematics at school, he decides to donate an amount of money on the following basis. He sets a mathematical quiz with 20 questions. For the correct answer to the first question (any learner may answer), the school will receive R 1, for a correct answer to the second question, the school will receive R 2, and so on. The donations $1; 2; 4; \ldots$ form a geometric sequence. Calculate, to the nearest Rand:
    a) The amount of money that the school will receive for the correct answer to the 20^{th} question.
    b) The total amount of money that the school will receive if all 20 questions are answered correctly.
14. The first term of a geometric sequence is 9, and the ratio of the sum of the first eight terms to the sum of the first four terms is $97 : 81$. Find the first three terms of the sequence, if it is given that all the terms are positive.
15. Given the geometric sequence: $6 + p; 10 + p; 15 + p$
    a) Determine p, ($p \neq -6$ or -10).
    b) Show that the constant ratio is $\frac{5}{4}$.
    c) Determine the tenth term of this sequence correct to one decimal place.
16. The second and fourth terms of a convergent geometric series are 36 and 16, respectively. Find the sum to infinity of this series, if all its terms are positive.
17. Evaluate:
$$\sum_{k=2}^{5} \frac{k(k+1)}{2}$$
18. $S_n = 4n^2 + 1$ represents the sum of the first n terms of a particular series. Find the second term.
19. Determine whether the following series converges for the given values of x. If it does converge, calculate the sum to infinity.
$$\sum_{p=1}^{\infty} (x+2)^p$$
    a) $x = -\frac{5}{2}$
    b) $x = -5$
20. Calculate:
$$\sum_{i=1}^{\infty} 5\left(4^{-i}\right)$$
21. The sum of the first p terms of a sequence is $p(p+1)$. Find the tenth term.
22. The powers of 2 are removed from the following set of positive integers
    $1; 2; 3; 4; 5; 6; \ldots; 1998; 1999; 2000$
    Find the sum of remaining integers.

23. Observe the pattern below:



  a) If the pattern continues, find the number of letters in the column containing M's.
  b) If the total number of letters in the pattern is 361, which letter will be in the last column.

24. Write $0{,}5\dot{7}$ as a proper fraction.
25. Given:

$$f(x) = \sum_{p=1}^{\infty} \frac{(1+x)^p}{1-x}$$

  a) For which values of x will $f(x)$ converge?
  b) Determine the value of $f\left(-\frac{1}{2}\right)$.

26. From the definition of a geometric sequence, deduce a formula for calculating the sum of n terms of the series

$$a^2 + a^4 + a^6 + \cdots$$

27. Calculate the tenth term of the series if $S_n = 2n + 3n^2$.
28. A theatre is filling up at a rate of 4 people in the first minute, 6 people in the second minute, and 8 people in the third minute and so on. After 6 minutes the theatre is half full. After how many minutes will the theatre be full?
[IEB, Nov 2001]
29. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 287Y	2a. 287Z	2b. 2882	3. 2883	4a. 2884	4b. 2885
5. 2886	6. 2887	7a. 2888	7b. 2889	8. 288B	9. 288C
10. 288D	11. 288F	12a. 288G	12b. 288H	13a. 288J	13b. 288K
14. 288M	15a. 288N	15b. 288P	15c. 288Q	16. 288R	17. 288S
18. 288T	19a. 288V	19b. 288W	20. 288X	21. 288Y	22. 288Z
23a. 2892	23b. 2893	24. 2894	25. 2895	26. 2896	27. 2897
28. 2898					

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