CHAPTER 3

Number patterns

- Discuss terminology.
- Emphasize the relationship between linear functions (general term) and linear sequences.
- Do not use the formula for arithmetic sequences.
- Emphasize the relationship between quadratic functions (general term) and quadratic sequences.
- Key activity in mathematical description of a pattern: finding the relationship between the number of the term and the value of the term.

3.1 Revision

Exercise 3 – 1: Linear sequences

1. Write down the next three terms in each of the following sequences: $45; 29; 13; -3; \ldots$

   Solution:

   $-19; -35; -51$

2. The general term is given for each sequence below. Calculate the missing terms.

   a) $-4; -9; -14; \ldots; -24$
   $T_n = 1 - 5n$

   b) $6; \ldots; 24; \ldots; 42$
   $T_n = 9n - 3$

   Solution:

   a)

   $$\begin{aligned} T_4 &= 1 - 5(4) \\ &= 1 - 20 \\ &= -19 \end{aligned}$$

   b)

   $$\begin{aligned} T_2 &= 9(2) - 3 \\ &= 18 - 3 \\ &= 15 \end{aligned}$$

   $$\begin{aligned} T_4 &= 9(4) - 3 \\ &= 36 - 3 \\ &= 33 \end{aligned}$$

3. Find the general formula for the following sequences and then find T_{10}, T_{15} and T_{30}:

a) $13; 16; 19; 22; \ldots$

b) $18; 24; 30; 36; \ldots$

c) $-10; -15; -20; -25; \ldots$

Solution:

a)

$$\begin{aligned} d &= T_2 - T_1 \\ &= 16 - 13 \\ &= 3 \\ \therefore T_n &= 10 + 3n \end{aligned}$$

$$\begin{aligned} T_n &= 10 + 3n \\ \therefore T_{10} &= 10 + 3(10) \\ &= 40 \\ \therefore T_{15} &= 10 + 3(15) \\ &= 55 \\ \therefore T_{30} &= 10 + 3(30) \\ &= 100 \end{aligned}$$

b)

$$\begin{aligned} d &= T_2 - T_1 \\ &= 24 - 18 \\ &= 6 \\ \therefore T_n &= 12 + 6n \end{aligned}$$

$$\begin{aligned} T_n &= 12 + 6n \\ \therefore T_{10} &= 12 + 6(10) \\ &= 72 \\ \therefore T_{15} &= 12 + 6(15) \\ &= 102 \\ \therefore T_{30} &= 12 + 6(30) \\ &= 192 \end{aligned}$$

c)

$$\begin{aligned} d &= T_2 - T_1 \\ &= -15 - (-10) \\ &= -5 \\ \therefore T_n &= -5 - 5n \end{aligned}$$

$$
\begin{aligned}
T_n &= -5 - 5n \\
\therefore T_{10} &= -5 - 5(10) \\
&= -55 \\
\therefore T_{15} &= -5 - 5(15) \\
&= -80 \\
\therefore T_{30} &= -5 - 5(30) \\
&= -155
\end{aligned}
$$

4. The seating in a classroom is arranged so that the first row has 20 desks, the second row has 22 desks, the third row has 24 desks and so on. Calculate how many desks are in the ninth row.

   Solution:

$$
\begin{aligned}
d &= T_2 - T_1 \\
&= 22 - 20 \\
&= 2 \\
\therefore T_n &= 18 + 2n
\end{aligned}
$$

$$
\begin{aligned}
T_n &= 18 + 2n \\
\therefore T_9 &= 18 + 2(9) \\
&= 18 + 18 \\
&= 36
\end{aligned}
$$

5. a) Complete the following:

$$
\begin{aligned}
13 + 31 &= \ldots \\
24 + 42 &= \ldots \\
38 + 83 &= \ldots
\end{aligned}
$$

   b) Look at the numbers on the left-hand side, what do you notice about the unit digit and the tens-digit?

   c) Investigate the pattern by trying other examples of 2-digit numbers.

   d) Make a conjecture about the pattern that you notice.

   e) Prove this conjecture.

   Solution:

   a)

$$
\begin{aligned}
13 + 31 &= 44 \\
24 + 42 &= 66 \\
38 + 83 &= 121
\end{aligned}
$$

   b) The unit digit and tens-digit have swapped position.

c)

$$\begin{aligned} 45 + 54 &= 99 \\ 71 + 17 &= 88 \end{aligned}$$

d) The sum of the two numbers will always be 11 times the sum of the two digits.

e) Let the first number be $a + 10b$ and let the second number be $b + 10a$:

$$\begin{aligned} \text{Number } 1 : &= a + 10b \\ \text{Number } 2 : &= b + 10a \\ \text{Number } 1 + 2 : &= a + b + 10a + 10b \\ &= 11a + 11b \\ &= 11(a + b) \end{aligned}$$

3.2 Quadratic sequences

Exercise 3 – 2: Quadratic sequences

1. Determine the second difference between the terms for the following sequences:

a) $5; 20; 45; 80; \ldots$

b) $6; 11; 18; 27; \ldots$

c) $1; 4; 9; 16; \ldots$

d) $3; 0; -5; -12; \ldots$

e) $1; 3; 7; 13; \ldots$

f) $0; -6; -16; -30; \ldots$

g) $-1; 2; 9; 20; \ldots$

h) $1; -3; -9; -17; \ldots$

i) $3a + 1; 12a + 1; 27a + 1; 48a + 1 \ldots$

j) $2; 10; 24; 44; \ldots$

k) $t - 2; 4t - 1; 9t; 16t + 1; \ldots$

Solution:

a)

$$\begin{aligned} \text{First differences: } &= 15; 25; 35 \\ \text{Second difference: } &= 10 \end{aligned}$$

b)

$$\begin{aligned} \text{First differences: } &= 5; 7; 9 \\ \text{Second difference: } &= 2 \end{aligned}$$

c)

$$\begin{aligned} \text{First differences: } &= 3; 5; 7 \\ \text{Second difference: } &= 2 \end{aligned}$$

d)

$$\begin{aligned} \text{First differences: } &= -3; -5; -7 \\ \text{Second difference: } &= -2 \end{aligned}$$

e)

$$\begin{aligned} \text{First differences: } &= 2; 4; 6 \\ \text{Second difference: } &= 2 \end{aligned}$$

f)

$$\begin{aligned} \text{First differences: } &= -6; -10; -14 \\ \text{Second difference: } &= -4 \end{aligned}$$

g)

$$\begin{aligned} \text{First differences: } &= 3; 7; 11 \\ \text{Second difference: } &= 4 \end{aligned}$$

h)

$$\begin{aligned} \text{First differences: } &= -4; -6; -8 \\ \text{Second difference: } &= -2 \end{aligned}$$

i)

$$\begin{aligned} \text{First differences: } &= 9a; 15a; 21a \\ \text{Second difference: } &= 6a \end{aligned}$$

j)

$$\begin{aligned} \text{First differences: } &= 8; 14; 20 \\ \text{Second difference: } &= 6 \end{aligned}$$

k)

$$\begin{aligned} \text{First differences: } &= 3t + 1; 5t + 1; 7t + 1 \\ \text{Second difference: } &= 2t \end{aligned}$$

2. Complete the sequence by filling in the missing term:

   a) $11; 21; 35; \ldots; 75$
   b) $20; \ldots; 42; 56; 72$
   c) $\ldots; 37; 65; 101$
   d) $3; \ldots; -13; -27; -45$
   e) $24; 35; 48; \ldots; 80$
   f) $\ldots; 11; 26; 47$

Solution:

a)

$$\begin{array}{lccccccc} & n=1 & & n=2 & & n=3 & & n=4 \\ T_n & 11 & & 21 & & 35 & & \mathbf{53} \\ 1^{\text{st}} \text{ difference} & & 10 & & 14 & & 18 & \\ 2^{\text{nd}} \text{ difference} & & & 4 & & 4 & & \end{array}$$

b)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	20		**30**		42		56
1st difference		10		12		14	
2nd difference			2		2		

c)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	**17**		37		65		101
1st difference		20		28		36	
2nd difference			8		8		

d)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	3		**-3**		-13		-27
1st difference		-6		-10		-14	
2nd difference			-4		-4		

e)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	24		35		48		**63**
1st difference		11		13		15	
2nd difference			2		2		

f)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	**2**		11		26		47
1st difference		9		15		21	
2nd difference			6		6		

3. Use the general term to generate the first four terms in each sequence:

   a) $T_n = n^2 + 3n - 1$
   b) $T_n = -n^2 - 5$
   c) $T_n = 3n^2 - 2n$
   d) $T_n = -2n^2 + n + 1$

Solution:

a) $3; 9; 17; 27$
b) $-6; -9; -14; -21$
c) $1; 8; 21; 40$
d) $0; -5; -14; -27$

Exercise 3 – 3: Quadratic sequences

1. Calculate the common second difference for each of the following quadratic sequences:

   a) $3;\ 6;\ 10;\ 15;\ 21;\ \ldots$

   b) $4;\ 9;\ 16;\ 25;\ 36;\ \ldots$

   c) $7;\ 17;\ 31;\ 49;\ 71;\ \ldots$

   d) $2;\ 10;\ 26;\ 50;\ 82;\ \ldots$

   e) $31;\ 30;\ 27;\ 22;\ 15;\ \ldots$

   Solution:

   a)
   $$\begin{aligned}\text{First differences: } &= 3; 4; 5; 6\\ \text{Second difference: } &= 1\end{aligned}$$

   b)
   $$\begin{aligned}\text{First differences: } &= 5; 7; 9; 11\\ \text{Second difference: } &= 2\end{aligned}$$

   c)
   $$\begin{aligned}\text{First differences: } &= 10; 14; 18; 22\\ \text{Second difference: } &= 4\end{aligned}$$

   d)
   $$\begin{aligned}\text{First differences: } &= 8; 16; 24; 32\\ \text{Second difference: } &= 8\end{aligned}$$

   e)
   $$\begin{aligned}\text{First differences: } &= -1; -3; -5; -7\\ \text{Second difference: } &= -2\end{aligned}$$

2. Find the first five terms of the quadratic sequence defined by: $T_n = 5n^2 + 3n + 4$.

   Solution:

   $$\begin{aligned}T_n &= 5n^2 + 3n + 4\\ T_1 &= 5(1)^2 + 3(1) + 4\\ &= 12\\ T_2 &= 5(2)^2 + 3(2) + 4\\ &= 30\\ T_3 &= 5(3)^2 + 3(3) + 4\\ &= 58\\ T_4 &= 5(4)^2 + 3(4) + 4\\ &= 96\\ T_5 &= 5(5)^2 + 3(5) + 4\\ &= 144\end{aligned}$$

   $12; 30; 58; 96; 144$

3. Given $T_n = 4n^2 + 5n + 10$, find T_9.

Solution:

$$
\begin{aligned}
T_n &= 4n^2 + 5n + 10 \\
T_9 &= 4(9)^2 + 5(9) + 10 \\
&= 379
\end{aligned}
$$

4. Given $T_n = 2n^2$, for which value of n does $T_n = 32$?

Solution:

$$
\begin{aligned}
T_n &= 2n^2 \\
32 &= 2n^2 \\
16 &= n^2 \\
4 &= n
\end{aligned}
$$

5. a) Write down the next two terms of the quadratic sequence: $16; 27; 42; 61; \ldots$

   b) Find the general formula for the quadratic sequence above.

Solution:

a)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	16		27		42		61
1st difference		11		15		19	
2nd difference			4		4		

$$
\begin{aligned}
\text{Second difference: } &= 4 \\
\text{First differences: } &= 11; 15; 19; 23; 27 \\
\therefore T_5 &= 61 + 23 \\
&= 84 \\
\therefore T_6 &= 84 + 27 \\
&= 111
\end{aligned}
$$

b)

$$
\begin{aligned}
T_n &= an^2 + bn + c \\
T_1 &= a + b + c \\
T_2 &= 4n^2 + 2n + c \\
T_3 &= 9n^2 + 3n + c
\end{aligned}
$$

$$\therefore a + b + c = 16$$
$$\therefore c = 16 - a - b$$
$$4a + 2b + c = 27$$
$$4a + 2b + (16 - a - b) = 27$$
$$3a + b = 11$$
$$\text{And } 9a + 3b + c = 42$$
$$\therefore 9a + 3b + (16 - a - b) = 42$$
$$8a + 2b = 26$$
$$4a + b = 13$$
$$\therefore b = 13 - 4a$$

$$\therefore 3a + b = 11$$
$$3a + (13 - 4a) = 11$$
$$-a = -2$$
$$\therefore a = 2$$
$$b = 13 - 4(2)$$
$$\therefore b = 5$$
$$\text{And } c = 16 - a - b$$
$$\therefore c = 16 - 2 - 5$$
$$= 9$$
$$\therefore T_n = 2n^2 + 5n + 9$$

3.3 Summary

Exercise 3 – 4: End of chapter exercises

1. Find the first five terms of the quadratic sequence defined by:
   $T_n = n^2 + 2n + 1$

   Solution:

   $-4; 9; 16; 25; 36$

2. Determine whether each of the following sequences is:

   - a linear sequence,
   - a quadratic sequence,
   - or neither.

a) $6; 9; 14; 21; 30; \ldots$

b) $1; 7; 17; 31; 49; \ldots$

c) $8; 17; 32; 53; 80; \ldots$

d) $9; 26; 51; 84; 125; \ldots$

e) $2; 20; 50; 92; 146; \ldots$

f) $5; 19; 41; 71; 109; \ldots$

g) $2; 6; 10; 14; 18; \ldots$

h) $3; 9; 15; 21; 27; \ldots$

i) $1; 2{,}5; 5; 8{,}5; 13; \ldots$

j) $10; 24; 44; 70; 102; \ldots$

k) $2\frac{1}{2}; 6; 10\frac{1}{2}; 16; 22\frac{1}{2}; \ldots$

l) $3p^2; 6p^2; 9p^2; 12p^2; 15p^2; \ldots$

m) $2k; 8k; 18k; 32k; 50k; \ldots$

Solution:

a)

$$\begin{aligned} \text{First differences: } &= 3; 5; 7; 9; \\ \text{Second difference: } &= 2 \end{aligned}$$

Quadratic sequence

b)

$$\begin{aligned} \text{First differences: } &= 6; 10; 14; 18 \\ \text{Second difference: } &= 4 \end{aligned}$$

Quadratic sequence

c)

$$\begin{aligned} \text{First differences: } &= 9; 15; 21; 27 \\ \text{Second difference: } &= 6 \end{aligned}$$

Quadratic sequence

d)

$$\begin{aligned} \text{First differences: } &= 17; 25; 33; 41 \\ \text{Second difference: } &= 8 \end{aligned}$$

Quadratic sequence

e)

$$\begin{aligned} \text{First differences: } &= 18; 30; 42; 54 \\ \text{Second difference: } &= 12 \end{aligned}$$

Quadratic sequence

f)

$$\begin{aligned} \text{First differences: } &= 14; 22; 30; 38 \\ \text{Second difference: } &= 8 \end{aligned}$$

Quadratic sequence

g)

$$\text{First difference: } = 4$$

Linear sequence

h)

$$\text{First difference: } = 6$$

Linear sequence

i)

$$\begin{aligned} \text{First differences: } &= 1{,}5;\ 2{,}5;\ 3{,}5;\ 4{,}5 \\ \text{Second difference: } &= 1 \end{aligned}$$

Quadratic sequence

j)

$$\begin{aligned} \text{First differences: } &= 14; 20; 26; 32 \\ \text{Second difference: } &= 16 \end{aligned}$$

Quadratic sequence

k)

$$\begin{aligned} \text{First differences: } &= 3{,}5;\ 4{,}5;\ 5{,}5;\ 6{,}5 \\ \text{Second difference: } &= 1 \end{aligned}$$

Quadratic sequence

l)

$$\text{First difference: } = 3p^2$$

Linear sequence

m)

$$\begin{aligned} \text{First differences: } &= 6k; 10k; 14k; 18k \\ \text{Second difference: } &= 4k \end{aligned}$$

Quadratic sequence

3. Given the pattern: $16; x; 46; \ldots$, determine the value of x if the pattern is linear.

   Solution:

$$\begin{aligned} x - 16 &= 46 - x \\ 2x &= 62 \\ \therefore x &= 31 \end{aligned}$$

4. Given $T_n = 2n^2$, for which value of n does $T_n = 242$?

   Solution:

$$\begin{aligned} 2n^2 &= 242 \\ n^2 &= 121 \\ \therefore n &= 11 \end{aligned}$$

5. Given $T_n = 3n^2$, find T_{11}.

Solution:

$$\begin{aligned} T_n &= 3n^2 \\ \therefore T_11 &= 3(11)^2 \\ &= 363 \end{aligned}$$

6. Given $T_n = n^2 + 4$, for which value of n does $T_n = 85$?

Solution:

$$\begin{aligned} n^2 + 4 &= 85 \\ n^2 &= 81 \\ \therefore n &= 9 \end{aligned}$$

7. Given $T_n = 4n^2 + 3n - 1$, find T_5.

Solution:

$$\begin{aligned} T_n &= 4n^2 + 3n - 1 \\ \therefore T_5 &= 4(5)^2 + 3(5) - 1 \\ &= 100 + 15 - 1 \\ &= 114 \end{aligned}$$

8. Given $T_n = \frac{3}{2}n^2$, for which value of n does $T_n = 96$?

Solution:

$$\begin{aligned} \frac{3}{2}n^2 &= 96 \\ n^2 &= 96 \times \frac{2}{3} \\ &= 64 \\ \therefore n &= 8 \end{aligned}$$

9. For each of the following patterns, determine:

- the next term in the pattern,
- and the general term,
- the tenth term in the pattern.

a) $3; 7; 11; 15; \ldots$

b) $17; 12; 7; 2; \ldots$

c) $\frac{1}{2}; 1; 1\frac{1}{2}; 2; \ldots$

d) $a; a + b; a + 2b; a + 3b; \ldots$

e) $1; -1; -3; -5; \ldots$

Solution:

a)

$$\begin{aligned}
d &= 7 - 3 \\
&= 4 \\
T_5 &= 15 + 4 \\
&= 19 \\
T_n &= 4n - 1 \\
T_{10} &= 4(10) - 1 \\
&= 39
\end{aligned}$$

b)

$$\begin{aligned}
d &= 12 - 17 \\
&= -5 \\
T_5 &= 2 - 5 \\
&= -3 \\
T_n &= 22 - 5n \\
T_{10} &= 22 - 5(10) \\
&= -28
\end{aligned}$$

c)

$$\begin{aligned}
d &= 1 - \frac{1}{2} \\
&= \frac{1}{2} \\
T_5 &= 2 + \frac{1}{2} \\
&= 2\frac{1}{2} \\
T_n &= \frac{1}{2}n \\
T_{10} &= \frac{1}{2}(10) \\
&= 5
\end{aligned}$$

d)

$$\begin{aligned}
d &= a + b - a \\
&= b \\
T_5 &= a + 3b + b \\
&= a + 4b \\
T_n &= a - b + bn \\
T_{10} &= a - b + b(10) \\
&= a + 9b
\end{aligned}$$

e)

$$
\begin{aligned}
d &= -1 - 1 \\
&= -2 \\
T_5 &= -5 - 2 \\
&= -7 \\
T_n &= 3 - 2n \\
T_{10} &= 3 - 2(10) \\
&= -17
\end{aligned}
$$

10. For each of the following sequences, find the equation for the general term and then use the equation to find T_{100}.

  a) $4; 7; 12; 19; 28; \ldots$
  b) $2; 8; 14; 20; 26; \ldots$
  c) $7; 13; 23; 37; 55; \ldots$
  d) $5; 14; 29; 50; 77; \ldots$

Solution:

a)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	4		7		12		19
1st difference		3		5		7	
2nd difference			2		2		

$$
\begin{aligned}
T_n &= an^2 + bn + c \\
\text{Second difference: } 2a &= 2 \\
\therefore a &= 1 \\
3a + b &= 3 \\
b &= 3 - 3(1) \\
\therefore b &= 0 \\
a + b + c &= 4 \\
\therefore c &= 4 - a - b \\
&= 4 - 1 \\
\therefore c &= 3 \\
\therefore T_n = n^2 + 3
\end{aligned}
$$

$$
\begin{aligned}
T_n &= n^2 + 3 \\
\therefore T_{100} &= (10)^2 + 3 \\
&= 10\ 003
\end{aligned}
$$

b)

$$
\begin{aligned}
d &= 8 - 2 \\
&= 6 \\
\therefore T_n &= 6n - 4
\end{aligned}
$$

$$\begin{aligned} T_n &= 6n - 4 \\ \therefore T_{100} &= 6(100) - 4 \\ &= 596 \end{aligned}$$

c)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	7		13		23		37
1st difference		6		10		14	
2nd difference			4		4		

$$\begin{aligned} T_n &= an^2 + bn + c \\ \text{Second difference: } 2a &= 4 \\ \therefore a &= 2 \\ 3a + b &= 6 \\ b &= 6 - 3(2) \\ \therefore b &= 0 \\ a + b + c &= 7 \\ \therefore c &= 7 - a - b \\ &= 7 - 2 \\ \therefore c &= 5 \end{aligned}$$

$$\therefore T_n = 2n^2 + 5$$

$$\begin{aligned} T_n &= 2n^2 + 5 \\ \therefore T_{100} &= 2(10)^2 + 5 \\ &= 20\ 005 \end{aligned}$$

d)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	5		14		29		50
1st difference		9		15		21	
2nd difference			6		6		

$$\begin{aligned} T_n &= an^2 + bn + c \\ \text{Second difference: } 2a &= 6 \\ \therefore a &= 3 \\ 3a + b &= 9 \\ b &= 9 - 3(3) \\ \therefore b &= 0 \\ a + b + c &= 5 \\ \therefore c &= 5 - a - b \\ &= 5 - 3 \\ \therefore c &= 2 \end{aligned}$$

$$\therefore T_n = 3n^2 + 2$$

$$
\begin{aligned}
T_n &= 3n^2 + 2 \\
\therefore T_{100} &= 3(10)^2 + 2 \\
&= 30\ 002
\end{aligned}
$$

11. Given: $T_n = 3n - 1$

    a) Write down the first five terms of the sequence.
    b) What do you notice about the difference between any two consecutive terms?
    c) Will this always be the case for a linear sequence?

    Solution:

    a) $2; 5; 8; 11; 14$
    b) Constant difference, $d = 3$
    c) Yes

12. Given the following sequence: $-15; -11; -7; \ldots; 173$

    a) Determine the equation for the general term.
    b) Calculate how many terms there are in the sequence.

    Solution:

    a)

$$
\begin{aligned}
d &= -11 - (-15) \\
&= 4 \\
T_n &= 4n - 19
\end{aligned}
$$

    b)

$$
\begin{aligned}
T_n &= 4n - 19 \\
173 &= 4n - 19 \\
192 &= 4n \\
\therefore 48 &= n
\end{aligned}
$$

13. Given $3; 7; 13; 21; 31; \ldots$

    a) Thabang determines that the general term is $T_n = 4n - 1$. Is he correct? Explain.
    b) Cristina determines that the general term is $T_n = n^2 + n + 1$. Is she correct? Explain.

    Solution:

    a)

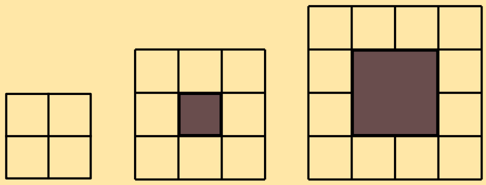
$$
\begin{aligned}
\text{First differences: } &= 4; 6; 8; 10 \\
\text{Second difference: } &= 2
\end{aligned}
$$

    Thabang's general term $T_n = 4n - 1$ describes a linear sequence and this is a quadratic sequence.

b)

$$
\begin{aligned}
T_n &= an^2 + bn + c \\
\text{Second difference: } 2a &= 2 \\
\therefore a &= 1 \\
3a + b &= 4 \\
b &= 4 - 3(1) \\
\therefore b &= 1 \\
a + b + c &= 3 \\
\therefore c &= 3 - a - b \\
&= 3 - 1 - 1 \\
\therefore c &= 1 \\
\therefore T_n &= n^2 + n + 1
\end{aligned}
$$

14. Given the following pattern of blocks:



2 3 4

a) Draw pattern 5.

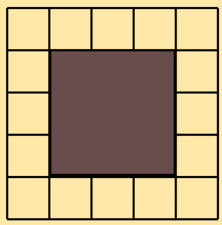
b) Complete the table below:

pattern number (n)	2	3	4	5	10	250	n
number of white blocks (w)	4	8					

c) Is this a linear or a quadratic sequence?

Solution:

a)



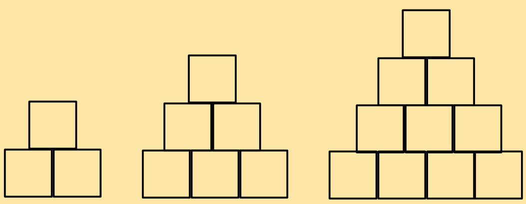
5

b)

pattern number (n)	2	3	4	5	10	250	n
number of white blocks (w)	4	8	12	16	36	996	$4n - 4$

Linear

15. Cubes of volume 1 cm^3 are stacked on top of each other to form a tower:



1 2 3

a) Complete the table for the height of the tower:

tower number (n)	1	2	3	4	10	n
height of tower (h)	2					

b) What type of sequence is this?

c) Now consider the number of cubes in each tower and complete the table below:

tower number (n)	1	2	3	4
number of cubes (c)	3			

d) What type of sequence is this?

e) Determine the general term for this sequence.

f) How many cubes are needed for tower number 21?

g) How high will a tower of 496 cubes be?

Solution:

a)

tower number (n)	1	2	3	4	10	n
height of tower (h)	2	3	4	5	11	$n+1$

b) Linear

c)

tower number (n)	1	2	3	4
number of cubes (c)	3	6	10	15

d) Quadratic

e)

$$\begin{aligned} T_n &= an^2 + bn + c \\ \text{Second difference: } 2a &= 1 \\ \therefore a &= \frac{1}{2} \\ 3a + b &= 3 \\ b &= 3 - 3(\frac{1}{2}) \\ \therefore b &= \frac{3}{2} \\ a + b + c &= 3 \\ \therefore c &= 3 - a - b \\ &= 3 - \frac{1}{2} - \frac{3}{2} \\ \therefore c &= 1 \\ \therefore T_n &= \frac{1}{2}n^2 + \frac{3}{2}n + 1 \end{aligned}$$

f)

$$\begin{aligned} T_n &= \frac{1}{2}n^2 + \frac{3}{2}n + 1 \\ \therefore T_{21} &= \frac{1}{2}(21)^2 + \frac{3}{2}(21) + 1 \\ &= \frac{441}{2} + \frac{63}{2} + \frac{2}{2} \\ &= 253 \end{aligned}$$

g)

$$\begin{aligned}
T_n &= \frac{1}{2}n^2 + \frac{3}{2}n + 1 \\
496 &= \frac{1}{2}n^2 + \frac{3}{2}n + 1 \\
0 &= \frac{1}{2}n^2 + \frac{3}{2}n - 495 \\
&= n^2 + 3n - 990 \\
&= (n-30)(n+33) \\
\therefore n = 30 &\text{ or } n = -33 \\
\therefore n &= 30 \\
\text{And } h &= n + 1 \\
&= 30 + 1 \\
&= 31 \text{ cm}
\end{aligned}$$

16. A quadratic sequence has a second term equal to 1, a third term equal to -6 and a fourth term equal to -14.

    a) Determine the second difference for this sequence.

    b) Hence, or otherwise, calculate the first term of the pattern.

    Solution:

    a)

$$\begin{aligned}
T_3 - T_2 &= -6 - (1) \\
&= -7 \\
T_4 - T_3 &= -14 - (-6) \\
&= -8 \\
\therefore \text{Second difference} &= -1
\end{aligned}$$

    b)

$$\begin{aligned}
T_1 &= 1 + 6 \\
&= 7
\end{aligned}$$

17. There are 15 schools competing in the U16 girls hockey championship and every team must play two matches — one home match and one away match.

    a) Use the given information to complete the table:

no. of schools	no. of matches
1	0
2	
3	
4	
5	

    b) Calculate the second difference.

    c) Determine a general term for the sequence.

    d) How many matches will be played if there are 15 schools competing in the championship?

e) If 600 matches must be played, how many schools are competing in the championship?

Solution:

a)

no. of schools	no. of matches
1	0
2	2
3	6
4	12
5	20

b)

$$\begin{aligned} T_2 - T_1 &= 2 - 0 \\ &= 2 \\ T_3 - T_2 &= 6 - 2 \\ &= 4 \\ T_4 - T_3 &= 12 - 6 \\ &= 6 \\ T_5 - T_4 &= 20 - 12 \\ &= 8 \\ \therefore \text{Second difference} &= 2 \end{aligned}$$

c)

$$\begin{aligned} T_n &= an^2 + bn + c \\ \text{Second difference: } 2a &= 2 \\ \therefore a &= 1 \\ 3a + b &= 2 \\ b &= 2 - 3(1) \\ \therefore b &= -1 \\ a + b + c &= 0 \\ \therefore c &= -a - b \\ &= -1 - (-1) \\ \therefore c &= 0 \\ \therefore T_n &= n^2 - n \end{aligned}$$

d)

$$\begin{aligned} T_n &= n^2 - n \\ T_{15} &= (15)^2 - 15 \\ &= 225 - 15 \\ &= 210 \end{aligned}$$

e)

$$\begin{aligned} T_n &= n^2 - n \\ 600 &= n^2 - n \\ 0 &= n^2 - n - 600 \\ &= (n-25)(n+24) \\ \therefore n &= 25 \text{ or } n = -24 \\ \therefore n &= 25 \end{aligned}$$

18. The first term of a quadratic sequence is 4, the third term is 34 and the common second difference is 10. Determine the first six terms in the sequence.

Solution:

$$\begin{aligned} \text{Let } T_2 &= x \\ \therefore T_2 - T_1 &= x - 4 \\ \text{And } T_3 - T_2 &= 34 - x \\ \text{Second difference } &= (T_3 - T_2) - (T_2 - T_1) \\ &= (34 - x) - (x - 4) \\ \therefore 10 &= 38 - 2x \\ 2x &= 28 \\ \therefore x &= 14 \end{aligned}$$

$4; 14; 34; 64; 104; 154$

19. Challenge question:

Given that the general term for a quadratic sequences is $T_n = an^2 + bn + c$, let d be the first difference and D be the second common difference.

a) Show that $a = \frac{D}{2}$.

b) Show that $b = d - \frac{3}{2}D$.

c) Show that $c = T_1 - d + D$.

d) Hence, show that $T_n = \frac{D}{2}n^2 + \left(d - \frac{3}{2}D\right)n + (T_1 - d + D)$.

Solution:

a)

$$\begin{aligned}
T_n &= an^2 + bn + c \\
T_1 &= a(1)^2 + b(1) + c \\
&= a + b + c \\
T_2 &= a(2)^2 + b(2) + c \\
&= 4a + 2b + c \\
T_3 &= a(3)^2 + b(3) + c \\
&= 9a + 6b + c \\
\text{First difference } d &= T_2 - T_1 \\
\therefore d &= (4a + 2b + c) - (a + b + c) \\
&= 3a + b \\
\therefore b &= d - 3a \\
\text{Second difference } D &= (T_3 - T_2) - (T_2 - T_1) \\
\therefore D &= (5a + b) - (3a + b) \\
&= 2a \\
\therefore a &= \frac{D}{2}
\end{aligned}$$

b)

$$\begin{aligned}
a &= \frac{D}{2} \\
b &= d - 3a \\
\therefore b &= d - 3\left(\frac{D}{2}\right) \\
&= d - \frac{3}{2}D
\end{aligned}$$

c)

$$\begin{aligned}
T_1 &= a + b + c \\
\therefore c &= T_1 - a - b \\
&= T_1 - \left(\frac{D}{2}\right) - \left(d - \frac{3}{2}D\right) \\
&= T_1 - \frac{D}{2} - d + \frac{3}{2}D \\
&= T_1 - d + D
\end{aligned}$$

d)

$$\begin{aligned}
T_n &= an^2 + bn + c \\
\therefore T_n &= \frac{D}{2}n^2 + \left(d - \frac{3}{2}D\right)n + (T_1 - d + D)
\end{aligned}$$