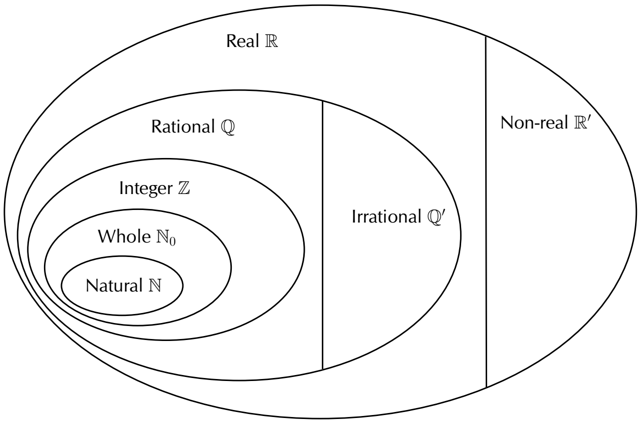
CHAPTER 1

Exponents and surds

1.1 Revision EMBF2

The number system EMBF3

The diagram below shows the structure of the number system:



See video: 2222 at www.everythingmaths.co.za

We use the following definitions:

- $\mathbb{N}$: natural numbers are $\{1;\ 2;\ 3;\ \ldots\}$
- $\mathbb{N}_0$: whole numbers are $\{0;\ 1;\ 2;\ 3;\ \ldots\}$
- $\mathbb{Z}$: integers are $\{\ldots;\ -3;\ -2;\ -1;\ 0;\ 1;\ 2;\ 3;\ \ldots\}$
- $\mathbb{Q}$: rational numbers are numbers which can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$, or as a terminating or recurring decimal number.
  Examples: $-\frac{7}{2}$; $-2{,}25$; 0; $\sqrt{9}$; $0{,}\dot{8}$; $\frac{23}{1}$
- $\mathbb{Q}'$: irrational numbers are numbers that cannot be written as a fraction with the numerator and denominator as integers. Irrational numbers also include decimal numbers that neither terminate nor recur.
  Examples: $\sqrt{3}$; $\sqrt[5]{2}$; π; $\frac{1+\sqrt{5}}{2}$; $1{,}27548\ldots$
- $\mathbb{R}$: real numbers include all rational and irrational numbers.
- $\mathbb{R}'$: non-real numbers or imaginary numbers are numbers that are not real.
  Examples: $\sqrt{-25}$; $\sqrt[4]{-1}$; $-\sqrt{-\frac{1}{16}}$

See video: 2223 at www.everythingmaths.co.za

Exercise 1 – 1: The number system

Use the list of words below to describe each of the following numbers (in some cases multiple words will be applicable):

- Natural ($\mathbb{N}$)
- Whole ($\mathbb{N}_0$)
- Integer ($\mathbb{Z}$)
- Rational ($\mathbb{Q}$)
- Irrational ($\mathbb{Q}'$)
- Real ($\mathbb{R}$)
- Non-real ($\mathbb{R}'$)

1. $\sqrt{7}$
2. 0,01
3. $16\frac{2}{5}$
4. $\sqrt{6\frac{1}{4}}$
5. 0
6. 2π
7. $-5{,}3\dot{8}$
8. $\frac{1-\sqrt{2}}{2}$
9. $-\sqrt{-3}$
10. $(\pi)^2$
11. $-\frac{9}{11}$
12. $\sqrt[3]{-8}$
13. $\frac{22}{7}$
14. 2,45897...
15. $0{,}\overline{65}$
16. $\sqrt[5]{-32}$

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1. 2224	2. 2225	3. 2226	4. 2227	5. 2228	6. 2229
7. 222B	8. 222C	9. 222D	10. 222F	11. 222G	12. 222H
13. 222J	14. 222K	15. 222M	16. 222N		

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Laws of exponents EMBF4

We use exponential notation to show that a number or variable is multiplied by itself a certain number of times. The exponent, also called the index or power, indicates the number of times the multiplication is repeated.

$$\text{base} \longleftarrow a^n \longrightarrow \text{exponent/index}$$

$$a^n = a \times a \times a \times \ldots \times a \quad (n \text{ times}) \qquad (a \in \mathbb{R}, n \in \mathbb{N})$$

See video: 222P at www.everythingmaths.co.za

Examples:

1. $2 \times 2 \times 2 \times 2 = 2^4$
2. $0{,}71 \times 0{,}71 \times 0{,}71 = (0{,}71)^3$
3. $(501)^2 = 501 \times 501$
4. $k^6 = k \times k \times k \times k \times k \times k$

For x^2, we say x is squared and for y^3, we say that y is cubed. In the last example we have k^6; we say that k is raised to the sixth power.

We also have the following definitions for exponents. It is important to remember that we always write the final answer with a positive exponent.

- $a^0 = 1$ ($a \neq 0$ because 0^0 is undefined)
- $a^{-n} = \frac{1}{a^n}$ ($a \neq 0$ because $\frac{1}{0}$ is undefined)

Examples:

1. $5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$
2. $(-36)^0 x = (1)x = x$
3. $\dfrac{7p^{-1}}{q^3 t^{-2}} = \dfrac{7t^2}{pq^3}$

We use the following laws for working with exponents:

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $(a^m)^n = a^{mn}$

where $a > 0$, $b > 0$ and $m, n \in \mathbb{Z}$.

Worked example 1: Laws of exponents

QUESTION

Simplify the following:

1. $5(m^{2t})^p \times 2(m^{3p})^t$
2. $\dfrac{8k^3x^2}{(xk)^2}$

3. $\dfrac{2^2 \times 3 \times 7^4}{(7 \times 2)^4}$

4. $3(3^b)^a$

SOLUTION

1. $5(m^{2t})^p \times 2(m^{3p})^t = 10m^{2pt+3pt} = 10m^{5pt}$
2. $\dfrac{8k^3x^2}{(xk)^2} = \dfrac{8k^3x^2}{x^2k^2} = 8k^{(3-2)}x^{(2-2)} = 8k^1x^0 = 8k$
3. $\dfrac{2^2 \times 3 \times 7^4}{(7 \times 2)^4} = \dfrac{2^2 \times 3 \times 7^4}{7^4 \times 2^4} = 2^{(2-4)} \times 3 \times 7^{(4-4)} = 2^{-2} \times 3 = \frac{3}{4}$
4. $3(3^b)^a = 3 \times 3^{ab} = 3^{ab+1}$

Worked example 2: Laws of exponents

QUESTION

Simplify: $\dfrac{3^m - 3^{m+1}}{4 \times 3^m - 3^m}$

SOLUTION

Step 1: Simplify to a form that can be factorised

$$\frac{3^m - 3^{m+1}}{4 \times 3^m - 3^m} = \frac{3^m - (3^m \times 3)}{4 \times 3^m - 3^m}$$

Step 2: Take out a common factor

$$= \frac{3^m(1-3)}{3^m(4-1)}$$

Step 3: Cancel the common factor and simplify

$$= \frac{1-3}{4-1}$$
$$= -\frac{2}{3}$$

Exercise 1 – 2: Laws of exponents

Simplify the following:

1. $4 \times 4^{2a} \times 4^2 \times 4^a$
2. $\dfrac{3^2}{2^{-3}}$
3. $(3p^5)^2$
4. $\dfrac{k^2k^{3x-4}}{k^x}$
5. $(5^{z-1})^2 + 5^z$
6. $(\frac{1}{4})^0$
7. $(x^2)^5$
8. $(\frac{a}{b})^{-2}$
9. $(m+n)^{-1}$
10. $2(p^t)^s$
11. $\dfrac{1}{(\frac{1}{a})^{-1}}$
12. $\frac{k^0}{k^{-1}}$
13. $\dfrac{-2}{-2^{-a}}$
14. $\dfrac{-h}{(-h)^{-3}}$
15. $\left(\dfrac{a^2b^3}{c^3d}\right)^2$
16. $10^7(7^0) \times 10^{-6}(-6)^0 - 6$
17. $m^3n^2 \div nm^2 \times \frac{mn}{2}$
18. $(2^{-2} - 5^{-1})^{-2}$
19. $(y^2)^{-3} \div \left(\frac{x^2}{y^3}\right)^{-1}$
20. $\dfrac{2^{c-5}}{2^{c-8}}$
21. $\dfrac{2^{9a} \times 4^{6a} \times 2^2}{8^{5a}}$
22. $\dfrac{20t^5p^{10}}{10t^4p^9}$
23. $\left(\dfrac{9q^{-2s}}{q^{-3s}y^{-4a-1}}\right)^2$

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1. 222R	2. 222S	3. 222T	4. 222V	5. 222W	6. 222X
7. 222Y	8. 222Z	9. 2232	10. 2233	11. 2234	12. 2235
13. 2236	14. 2237	15. 2238	16. 2239	17. 223B	18. 223C
19. 223D	20. 223F	21. 223G	22. 223H	23. 223J	

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See video: 222Q at www.everythingmaths.co.za

1.2 Rational exponents and surds EMBF5

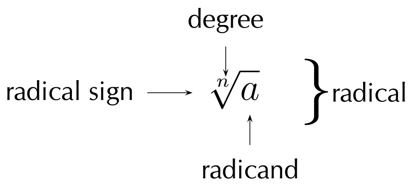
The laws of exponents can also be extended to include the rational numbers. A rational number is any number that can be written as a fraction with an integer in the numerator and in the denominator. We also have the following definitions for working with rational exponents.

- If $r^n = a$, then $r = \sqrt[n]{a} \quad (n \geq 2)$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{-\frac{1}{n}} = (a^{-1})^{\frac{1}{n}} = \sqrt[n]{\frac{1}{a}}$
- $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

where $a > 0$, $r > 0$ and $m, n \in \mathbb{Z}$, $n \neq 0$.

For $\sqrt{25} = 5$, we say that 5 is the square root of 25 and for $\sqrt[3]{8} = 2$, we say that 2 is the cube root of 8. For $\sqrt[5]{32} = 2$, we say that 2 is the fifth root of 32.

When dealing with exponents, a root refers to a number that is repeatedly multiplied by itself a certain number of times to get another number. A radical refers to a number written as shown below.



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The radical symbol and degree show which root is being determined. The radicand is the number under the radical symbol.

- If n is an even natural number, then the radicand must be positive, otherwise the roots are not real. For example, $\sqrt[4]{16} = 2$ since $2 \times 2 \times 2 \times 2 = 16$, but the roots of $\sqrt[4]{-16}$ are not real since $(-2) \times (-2) \times (-2) \times (-2) \neq -16$.
- If n is an odd natural number, then the radicand can be positive or negative. For example, $\sqrt[3]{27} = 3$ since $3 \times 3 \times 3 = 27$ and we can also determine $\sqrt[3]{-27} = -3$ since $(-3) \times (-3) \times (-3) = -27$.

It is also possible for there to be more than one n^{th} root of a number. For example, $(-2)^2 = 4$ and $2^2 = 4$, so both -2 and 2 are square roots of 4.

A surd is a radical which results in an irrational number. Irrational numbers are numbers that cannot be written as a fraction with the numerator and the denominator as integers. For example, $\sqrt{12}$, $\sqrt[3]{100}$, $\sqrt[5]{25}$ are surds.

Worked example 3: Rational exponents

QUESTION

Write each of the following as a radical and simplify where possible:

1. $18^{\frac{1}{2}}$
2. $(-125)^{-\frac{1}{3}}$

3. $4^{\frac{3}{2}}$
4. $(-81)^{\frac{1}{2}}$
5. $(0{,}008)^{\frac{1}{3}}$

SOLUTION

1. $18^{\frac{1}{2}} = \sqrt{18}$
2. $(-125)^{-\frac{1}{3}} = \sqrt[3]{(-125)^{-1}} = \sqrt[3]{\frac{1}{-125}} = \sqrt[3]{\frac{1}{(-5)^3}} = -\frac{1}{5}$
3. $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = \sqrt{4^3} = \sqrt{64} = 8$
4. $(-81)^{\frac{1}{2}} = \sqrt{-81} = \text{not real}$
5. $(0{,}008)^{\frac{1}{3}} = \sqrt[3]{\frac{8}{1000}} = \sqrt[3]{\frac{2^3}{10^3}} = \frac{2}{10} = \frac{1}{5}$

See video: 223M at www.everythingmaths.co.za

Worked example 4: Rational exponents

QUESTION

Simplify without using a calculator:

$$\left(\frac{5}{4^{-1} - 9^{-1}}\right)^{\frac{1}{2}}$$

SOLUTION

Step 1: Write the fraction with positive exponents in the denominator

$$\left(\frac{5}{\frac{1}{4} - \frac{1}{9}}\right)^{\frac{1}{2}}$$

Step 2: Simplify the denominator

$$= \left(\frac{5}{\frac{9-4}{36}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{5}{\frac{5}{36}}\right)^{\frac{1}{2}}$$

$$= \left(5 \div \frac{5}{36}\right)^{\frac{1}{2}}$$

$$= \left(5 \times \frac{36}{5}\right)^{\frac{1}{2}}$$

$$= (36)^{\frac{1}{2}}$$

Step 3: Take the square root

$$= \sqrt{36}$$

$$= 6$$

Exercise 1 – 3: Rational exponents and surds

1. Simplify the following and write answers with positive exponents:

   a) $\sqrt{49}$
   b) $\sqrt{36^{-1}}$
   c) $\sqrt[3]{6^{-2}}$
   d) $\sqrt[3]{-\frac{64}{27}}$
   e) $\sqrt[4]{(16x^4)^3}$

2. Simplify:

   a) $s^{\frac{1}{2}} \div s^{\frac{1}{3}}$
   b) $\left(64m^6\right)^{\frac{2}{3}}$
   c) $\dfrac{12m^{\frac{7}{9}}}{8m^{-\frac{11}{9}}}$
   d) $(5x)^0 + 5x^0 - (0{,}25)^{-0{,}5} + 8^{\frac{2}{3}}$

3. Use the laws to re-write the following expression as a power of x:

   $x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}$

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1a. 223P 1b. 223Q 1c. 223R 1d. 223S 1e. 223T 2a. 223V
2b. 223W 2c. 223X 2d. 223Y 3. 223Z

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Simplification of surds EMBF6

We have seen in previous examples and exercises that rational exponents are closely related to surds. It is often useful to write a surd in exponential notation as it allows us to use the exponential laws.

The additional laws listed below make simplifying surds easier:

- $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$
- $\sqrt[n]{a^m} = a^{\frac{m}{n}}$
- $\left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}$

See video: 223N at www.everythingmaths.co.za

Worked example 5: Simplifying surds

QUESTION

Show that:

1. $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

SOLUTION

1.

$$\begin{aligned}\sqrt[n]{a} \times \sqrt[n]{b} &= a^{\frac{1}{n}} \times b^{\frac{1}{n}}\\ &= (ab)^{\frac{1}{n}}\\ &= \sqrt[n]{ab}\end{aligned}$$

2.

$$\begin{aligned}\sqrt[n]{\frac{a}{b}} &= \left(\frac{a}{b}\right)^{\frac{1}{n}}\\ &= \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}\\ &= \frac{\sqrt[n]{a}}{\sqrt[n]{b}}\end{aligned}$$

Examples:

1. $\sqrt{2} \times \sqrt{32} = \sqrt{2 \times 32} = \sqrt{64} = 8$
2. $\frac{\sqrt[3]{24}}{\sqrt[3]{3}} = \sqrt[3]{\frac{24}{3}} = \sqrt[3]{8} = 2$
3. $\sqrt{\sqrt{81}} = \sqrt[4]{81} = \sqrt[4]{3^4} = 3$

Like and unlike surds EMBF7

Two surds $\sqrt[m]{a}$ and $\sqrt[n]{b}$ are like surds if $m = n$, otherwise they are called unlike surds. For example, $\sqrt{\frac{1}{3}}$ and $-\sqrt{61}$ are like surds because $m = n = 2$. Examples of unlike surds are $\sqrt[3]{5}$ and $\sqrt[5]{7y^3}$ since $m \neq n$.

Simplest surd form EMBF8

We can sometimes simplify surds by writing the radicand as a product of factors that can be further simplified using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

See video: 2242 at www.everythingmaths.co.za

Worked example 6: Simplest surd form

QUESTION

Write the following in simplest surd form: $\sqrt{50}$

SOLUTION

Step 1: Write the radicand as a product of prime factors

$$
\begin{aligned}
\sqrt{50} &= \sqrt{5 \times 5 \times 2} \\
&= \sqrt{5^2 \times 2}
\end{aligned}
$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$
\begin{aligned}
&= \sqrt{5^2} \times \sqrt{2} \\
&= 5 \times \sqrt{2} \\
&= 5\sqrt{2}
\end{aligned}
$$

Sometimes a surd cannot be simplified. For example, $\sqrt{6}$, $\sqrt[3]{30}$ and $\sqrt[4]{42}$ are already in their simplest form.

Worked example 7: Simplest surd form

QUESTION

Write the following in simplest surd form: $\sqrt[3]{54}$

SOLUTION

Step 1: Write the radicand as a product of prime factors

$$\begin{aligned}\sqrt[3]{54} &= \sqrt[3]{3 \times 3 \times 3 \times 2} \\ &= \sqrt[3]{3^3 \times 2}\end{aligned}$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$\begin{aligned} &= \sqrt[3]{3^3} \times \sqrt[3]{2} \\ &= 3 \times \sqrt[3]{2} \\ &= 3\sqrt[3]{2}\end{aligned}$$

Worked example 8: Simplest surd form

QUESTION

Simplify: $\sqrt{147} + \sqrt{108}$

SOLUTION

Step 1: Write the radicands as a product of prime factors

$$\begin{aligned}\sqrt{147} + \sqrt{108} &= \sqrt{49 \times 3} + \sqrt{36 \times 3} \\ &= \sqrt{7^2 \times 3} + \sqrt{6^2 \times 3}\end{aligned}$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$
\begin{aligned}
&= \left(\sqrt{7^2} \times \sqrt{3}\right) + \left(\sqrt{6^2} \times \sqrt{3}\right) \\
&= \left(7 \times \sqrt{3}\right) + \left(6 \times \sqrt{3}\right) \\
&= 7\sqrt{3} + 6\sqrt{3}
\end{aligned}
$$

Step 3: Simplify and write the final answer

$$13\sqrt{3}$$

Worked example 9: Simplest surd form

QUESTION

Simplify: $\left(\sqrt{20} - \sqrt{5}\right)^2$

SOLUTION

Step 1: Factorise the radicands were possible

$$\left(\sqrt{20} - \sqrt{5}\right)^2 = \left(\sqrt{4 \times 5} - \sqrt{5}\right)^2$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$
\begin{aligned}
&= \left(\sqrt{4} \times \sqrt{5} - \sqrt{5}\right)^2 \\
&= \left(2 \times \sqrt{5} - \sqrt{5}\right)^2 \\
&= \left(2\sqrt{5} - \sqrt{5}\right)^2
\end{aligned}
$$

Step 3: Simplify and write the final answer

$$
\begin{aligned}
&= \left(\sqrt{5}\right)^2 \\
&= 5
\end{aligned}
$$

Worked example 10: Simplest surd form with fractions

QUESTION

Write in simplest surd form: $\sqrt{75} \times \sqrt[3]{(48)^{-1}}$

SOLUTION

Step 1: Factorise the radicands were possible

$$\begin{aligned}
\sqrt{75} \times \sqrt[3]{(48)^{-1}} &= \sqrt{25 \times 3} \times \sqrt[3]{\frac{1}{48}} \\
&= \sqrt{25 \times 3} \times \frac{1}{\sqrt[3]{8 \times 6}}
\end{aligned}$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$\begin{aligned}
&= \sqrt{25} \times \sqrt{3} \times \frac{1}{\sqrt[3]{8} \times \sqrt[3]{6}} \\
&= 5 \times \sqrt{3} \times \frac{1}{2 \times \sqrt[3]{6}}
\end{aligned}$$

Step 3: Simplify and write the final answer

$$\begin{aligned}
&= 5\sqrt{3} \times \frac{1}{2\sqrt[3]{6}} \\
&= \frac{5\sqrt{3}}{2\sqrt[3]{6}}
\end{aligned}$$

Exercise 1 – 4: Simplification of surds

1. Simplify the following and write answers with positive exponents:

   a) $\sqrt[3]{16} \times \sqrt[3]{4}$

   b) $\sqrt{a^2b^3} \times \sqrt{b^5c^4}$

   c) $\dfrac{\sqrt{12}}{\sqrt{3}}$

   d) $\sqrt{x^2y^{13}} \div \sqrt{y^5}$

2. Simplify the following:

   a) $\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$

   b) $\frac{b-a}{a^{\frac{1}{2}} - b^{\frac{1}{2}}}$

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1a. 2243 1b. 2244 1c. 2245 1d. 2246 2a. 2247 2b. 2248

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Rationalising denominators EMBF9

It is often easier to work with fractions that have rational denominators instead of surd denominators. By rationalising the denominator, we convert a fraction with a surd in the denominator to a fraction that has a rational denominator.

Worked example 11: Rationalising the denominator

QUESTION

Rationalise the denominator:

$$\frac{5x-16}{\sqrt{x}}$$

SOLUTION

Step 1: Multiply the fraction by $\frac{\sqrt{x}}{\sqrt{x}}$

Notice that $\frac{\sqrt{x}}{\sqrt{x}} = 1$, so the value of the fraction has not been changed.

$$\frac{5x-16}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}(5x-16)}{\sqrt{x} \times \sqrt{x}}$$

Step 2: Simplify the denominator

$$= \frac{\sqrt{x}(5x-16)}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x}(5x-16)}{x}$$

The term in the denominator has changed from a surd to a rational number. Expressing the surd in the numerator is the preferred way of writing expressions.

Worked example 12: Rationalising the denominator

QUESTION

Write the following with a rational denominator:

$$\frac{y-25}{\sqrt{y}+5}$$

SOLUTION

Step 1: Multiply the fraction by $\frac{\sqrt{y}-5}{\sqrt{y}-5}$

To eliminate the surd from the denominator, we must multiply the fraction by an expression that will result in a difference of two squares in the denominator.

$$\frac{y-25}{\sqrt{y}+5} \times \frac{\sqrt{y}-5}{\sqrt{y}-5}$$

Step 2: Simplify the denominator

$$\begin{aligned} &= \frac{(y-25)(\sqrt{y}-5)}{(\sqrt{y}+5)(\sqrt{y}-5)} \\ &= \frac{(y-25)(\sqrt{y}-5)}{(\sqrt{y})^2-25} \\ &= \frac{(y-25)(\sqrt{y}-5)}{y-25} \\ &= \sqrt{y}-5 \end{aligned}$$

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Exercise 1 – 5: Rationalising the denominator

Rationalise the denominator in each of the following:

1. $\frac{10}{\sqrt{5}}$
2. $\frac{3}{\sqrt{6}}$
3. $\frac{2}{\sqrt{3}} \div \frac{\sqrt{2}}{3}$
4. $\frac{3}{\sqrt{5}-1}$
5. $\frac{x}{\sqrt{y}}$
6. $\frac{\sqrt{3}+\sqrt{7}}{\sqrt{2}}$
7. $\frac{3\sqrt{p}-4}{\sqrt{p}}$
8. $\frac{t-4}{\sqrt{t}+2}$
9. $(1+\sqrt{m})^{-1}$
10. $a\left(\sqrt{a} \div \sqrt{b}\right)^{-1}$

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1. 224B 2. 224C 3. 224D 4. 224F 5. 224G 6. 224H
7. 224J 8. 224K 9. 224M 10. 224N

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1.3 Solving surd equations EMBFB

We also need to be able to solve equations that involve surds.

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Worked example 13: Surd equations

QUESTION

Solve for x: $5\sqrt[3]{x^4} = 405$

SOLUTION

Step 1: Write in exponential notation

$$\begin{aligned} 5\left(x^4\right)^{\frac{1}{3}} &= 405 \\ 5x^{\frac{4}{3}} &= 405 \end{aligned}$$

Step 2: Divide both sides of the equation by 5 and simplify

$$\begin{aligned} \frac{5x^{\frac{4}{3}}}{5} &= \frac{405}{5} \\ x^{\frac{4}{3}} &= 81 \\ x^{\frac{4}{3}} &= 3^4 \end{aligned}$$

Step 3: Simplify the exponents

$$\begin{aligned} \left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} &= \left(3^4\right)^{\frac{3}{4}} \\ x &= 3^3 \\ x &= 27 \end{aligned}$$

Step 4: Check the solution by substituting the answer back into the original equation

$$\begin{aligned} \text{LHS} &= 5\sqrt[3]{x^4} \\ &= 5(27)^{\frac{4}{3}} \\ &= 5(3^3)^{\frac{4}{3}} \\ &= 5(3^4) \\ &= 405 \\ &= \text{RHS} \end{aligned}$$

Worked example 14: Surd equations

QUESTION

Solve for z: $z - 4\sqrt{z} + 3 = 0$

SOLUTION

Step 1: Factorise

$$\begin{aligned} z - 4\sqrt{z} + 3 &= 0 \\ z - 4z^{\frac{1}{2}} + 3 &= 0 \\ (z^{\frac{1}{2}} - 3)(z^{\frac{1}{2}} - 1) &= 0 \end{aligned}$$

Step 2: Solve for both factors

The zero law states: if $a \times b = 0$, then $a = 0$ or $b = 0$.

$$\therefore (z^{\frac{1}{2}} - 3) = 0 \text{ or } (z^{\frac{1}{2}} - 1) = 0$$

Therefore

$$\begin{aligned} z^{\frac{1}{2}} - 3 &= 0 \\ z^{\frac{1}{2}} &= 3 \\ \left(z^{\frac{1}{2}}\right)^2 &= 3^2 \\ z &= 9 \end{aligned}$$

or

$$
\begin{aligned}
z^{\frac{1}{2}} - 1 &= 0 \\
z^{\frac{1}{2}} &= 1 \\
\left(z^{\frac{1}{2}}\right)^2 &= 1^2 \\
z &= 1
\end{aligned}
$$

Step 3: Check the solution by substituting both answers back into the original equation

If $z = 9$:

$$
\begin{aligned}
\text{LHS} &= z - 4\sqrt{z} + 3 \\
&= 9 - 4\sqrt{9} + 3 \\
&= 12 - 12 \\
&= 0 \\
&= \text{RHS}
\end{aligned}
$$

If $z = 1$:

$$
\begin{aligned}
\text{LHS} &= z - 4\sqrt{z} + 3 \\
&= 1 - 4\sqrt{1} + 3 \\
&= 4 - 4 \\
&= 0 \\
&= \text{RHS}
\end{aligned}
$$

Step 4: Write the final answer

The solution to $z - 4\sqrt{z} + 3 = 0$ is $z = 9$ or $z = 1$.

Worked example 15: Surd equations

QUESTION

Solve for p: $\sqrt{p-2} - 3 = 0$

SOLUTION

Step 1: Write the equation with only the square root on the left hand side

Use the additive inverse to get all other terms on the right hand side and only the

square root on the left hand side.

$$\sqrt{p-2} = 3$$

Step 2: Square both sides of the equation

$$\begin{aligned}\left(\sqrt{p-2}\right)^2 &= 3^2\\ p-2 &= 9\\ p &= 11\end{aligned}$$

Step 3: Check the solution by substituting the answer back into the original equation

If $p = 11$:

$$\begin{aligned}\text{LHS} &= \sqrt{p-2} - 3\\ &= \sqrt{11-2} - 3\\ &= \sqrt{9} - 3\\ &= 3 - 3\\ &= 0\\ &= \text{RHS}\end{aligned}$$

Step 4: Write the final answer

The solution to $\sqrt{p-2} - 3 = 0$ is $p = 11$.

Exercise 1 – 6: Solving surd equations

Solve for the unknown variable (remember to check that the solution is valid):

1. $2^{x+1} - 32 = 0$
2. $125\left(3^p\right) = 27\left(5^p\right)$
3. $2y^{\frac{1}{2}} - 3y^{\frac{1}{4}} + 1 = 0$
4. $t - 1 = \sqrt{7-t}$
5. $2z - 7\sqrt{z} + 3 = 0$
6. $x^{\frac{1}{3}}(x^{\frac{1}{3}} + 1) = 6$
7. $2^{4n} - \frac{1}{\sqrt[4]{16}} = 0$
8. $\sqrt{31 - 10d} = 4 - d$
9. $y - 10\sqrt{y} + 9 = 0$
10. $f = 2 + \sqrt{19 - 2f}$

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1. 224Q 2. 224R 3. 224S 4. 224T 5. 224V 6. 224W
7. 224X 8. 224Y 9. 224Z 10. 2252

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1.4 Applications of exponentials EMBFC

There are many real world applications that require exponents. For example, exponentials are used to determine population growth and they are also used in finance to calculate different types of interest.

Worked example 16: Applications of exponentials

QUESTION

A type of bacteria has a very high exponential growth rate at 80% every hour. If there are 10 bacteria, determine how many there will be in five hours, in one day and in one week?

SOLUTION

Step 1: Exponential formula

$$\text{final population} = \text{initial population} \times (1 + \text{growth percentage})^{\text{time period in hours}}$$

Therefore, in this case:

$$\text{final population} = 10\,(1{,}8)^{n}$$

where n = number of hours.

Step 2: In 5 hours

$$\text{final population} = 10\,(1{,}8)^{5} \approx 189$$

Step 3: In 1 day = 24 hours

$$\text{final population} = 10\,(1{,}8)^{24} \approx 13\ 382\ 588$$

Step 4: In 1 week = 168 hours

$$\text{final population} = 10\,(1{,}8)^{168} \approx 7{,}687 \times 10^{43}$$

Note this answer is given in scientific notation as it is a very big number.

Worked example 17: Applications of exponentials

QUESTION

A species of extremely rare deep water fish has a very long lifespan and rarely has offspring. If there are a total of 821 of this type of fish and their growth rate is 2% each month, how many will there be in half of a year? What will the population be in ten years and in one hundred years?

SOLUTION

Step 1: Exponential formula

$$\text{final population} = \text{initial population} \times (1 + \text{growth percentage})^{\text{time period in months}}$$

Therefore, in this case:

$$\text{final population} = 821(1{,}02)^{n}$$

where n = number of months.

Step 2: In half a year = 6 months

final population $= 821(1{,}02)^{6} \approx 925$

Step 3: In 10 years = 120 months

final population $= 821(1{,}02)^{120} \approx 8838$

Step 4: In 100 years = 1200 months

final population $= 821(1{,}02)^{1200} \approx 1{,}716 \times 10^{13}$

Note this answer is also given in scientific notation as it is a very big number.

Exercise 1 – 7: Applications of exponentials

1. Nqobani invests R 5530 into an account which pays out a lump sum at the end of 6 years. If he gets R 9622,20 at the end of the period, what compound interest rate did the bank offer him? Give answer correct to one decimal place.
2. The current population of Johannesburg is 3 885 840 and the average rate of population growth in South Africa is 0,7% p.a. What can city planners expect the population of Johannesburg to be in 13 years time?
3. Abiona places 3 books in a stack on her desk. The next day she counts the books in the stack and then adds the same number of books to the top of the stack. After how many days will she have a stack of 192 books?

4. A type of mould has a very high exponential growth rate of 40% every hour. If there are initially 45 individual mould cells in the population, determine how many there will be in 19 hours.

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1. 2253 2. 2254 3. 2255 4. 2256

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1.5 Summary EMBFD

See presentation: 2257 at www.everythingmaths.co.za

1. The number system:
   - $\mathbb{N}$: natural numbers are $\{1;\ 2;\ 3;\ \ldots\}$
   - $\mathbb{N}_0$: whole numbers are $\{0;\ 1;\ 2;\ 3;\ \ldots\}$
   - $\mathbb{Z}$: integers are $\{\ldots;\ -3;\ -2;\ -1;\ 0;\ 1;\ 2;\ 3;\ \ldots\}$
   - $\mathbb{Q}$: rational numbers are numbers which can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$, or as a terminating or recurring decimal number.
   - $\mathbb{Q}'$: irrational numbers are numbers that cannot be written as a fraction with the numerator and denominator as integers. Irrational numbers also include decimal numbers that neither terminate nor recur.
   - $\mathbb{R}$: real numbers include all rational and irrational numbers.
   - $\mathbb{R}'$: non-real numbers or imaginary numbers are numbers that are not real.
2. Definitions:
   - $a^n = a \times a \times a \times \cdots \times a \quad (n \text{ times}) \qquad (a \in \mathbb{R}, n \in \mathbb{N})$
   - $a^0 = 1 \qquad (a \neq 0$ because 0^0 is undefined)
   - $a^{-n} = \frac{1}{a^n} \qquad (a \neq 0$ because $\dfrac{1}{0}$ is undefined)
3. Laws of exponents:
   - $a^m \times a^n = a^{m+n}$
   - $\dfrac{a^m}{a^n} = a^{m-n}$
   - $(ab)^n = a^n b^n$
   - $\left(\frac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
   - $(a^m)^n = a^{mn}$

   where $a > 0$, $b > 0$ and $m, n \in \mathbb{Z}$.

4. Rational exponents and surds:

   - If $r^n = a$, then $r = \sqrt[n]{a} \quad (n \geq 2)$
   - $a^{\frac{1}{n}} = \sqrt[n]{a}$
   - $a^{-\frac{1}{n}} = (a^{-1})^{\frac{1}{n}} = \sqrt[n]{\frac{1}{a}}$
   - $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

   where $a > 0$, $r > 0$ and $m, n \in \mathbb{Z}$, $n \neq 0$.

5. Simplification of surds:

   - $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$
   - $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
   - $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Exercise 1 – 8: End of chapter exercises

1. Simplify as far as possible:

   a) $8^{-\frac{2}{3}}$

   b) $\sqrt{16} + 8^{-\frac{2}{3}}$

2. Simplify:

   a) $\left(x^3\right)^{\frac{4}{3}}$

   b) $\left(s^2\right)^{\frac{1}{2}}$

   c) $\left(m^5\right)^{\frac{5}{3}}$

   d) $\left(-m^2\right)^{\frac{4}{3}}$

   e) $-\left(m^2\right)^{\frac{4}{3}}$

   f) $\left(3y^{\frac{4}{3}}\right)^4$

3. Simplify the following:

   a) $\frac{3a^{-2}b^{15}c^{-5}}{(a^{-4}b^3c)^{\frac{-5}{2}}}$

   b) $\left(9a^6b^4\right)^{\frac{1}{2}}$

   c) $\left(a^{\frac{3}{2}}b^{\frac{3}{4}}\right)^{16}$

   d) $x^3\sqrt{x}$

   e) $\sqrt[3]{x^4b^5}$

4. Re-write the following expression as a power of x:

   $$\frac{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}}{x^2}$$

5. Expand:

   $$\left(\sqrt{x} - \sqrt{2}\right)\left(\sqrt{x} + \sqrt{2}\right)$$

6. Rationalise the denominator:

   $$\frac{10}{\sqrt{x} - \frac{1}{x}}$$

7. Write as a single term with a rational denominator:
$\frac{3}{2\sqrt{x}} + \sqrt{x}$

8. Write in simplest surd form:

a) $\sqrt{72}$

b) $\sqrt{45} + \sqrt{80}$

c) $\frac{\sqrt{48}}{\sqrt{12}}$

d) $\frac{\sqrt{18} \div \sqrt{72}}{\sqrt{8}}$

e) $\frac{4}{(\sqrt{8} \div \sqrt{2})}$

f) $\frac{16}{(\sqrt{20} \div \sqrt{12})}$

9. Expand and simplify:

a) $\left(2 + \sqrt{2}\right)^2$

b) $\left(2 + \sqrt{2}\right)\left(1 + \sqrt{8}\right)$

c) $\left(1 + \sqrt{3}\right)\left(1 + \sqrt{8} + \sqrt{3}\right)$

10. Simplify, without use of a calculator:

a) $\sqrt{5}\left(\sqrt{45} + 2\sqrt{80}\right)$

b) $\frac{\sqrt{98} - \sqrt{8}}{\sqrt{50}}$

11. Simplify:
$\sqrt{98x^6} + \sqrt{128x^6}$

12. Rationalise the denominator:

a) $\frac{\sqrt{5} + 2}{\sqrt{5}}$

b) $\frac{y - 4}{\sqrt{y} - 2}$

c) $\frac{2x - 20}{\sqrt{x} - \sqrt{10}}$

13. Evaluate without using a calculator: $\left(2 - \frac{\sqrt{7}}{2}\right)^{\frac{1}{2}} \times \left(2 + \frac{\sqrt{7}}{2}\right)^{\frac{1}{2}}$

14. Prove (without the use of a calculator):
$\sqrt{\frac{8}{3}} + 5\sqrt{\frac{5}{3}} - \sqrt{\frac{1}{6}} = \frac{10\sqrt{15} + 3\sqrt{6}}{6}$

15. Simplify completely by showing all your steps (do not use a calculator):

$$3^{-\frac{1}{2}}\left[\sqrt{12} + \sqrt[3]{\left(3\sqrt{3}\right)}\right]$$

16. Fill in the blank surd-form number on the right hand side of the equal sign which will make the following a true statement: $-3\sqrt{6} \times -2\sqrt{24} = -\sqrt{18} \times \ldots$

17. Solve for the unknown variable:

    a) $3^{x-1} - 27 = 0$

    b) $8^{x} - \frac{1}{\sqrt[3]{8}} = 0$

    c) $27(4^{x}) = (64)3^{x}$

    d) $\sqrt{2x-5} = 2 - x$

    e) $2x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 2 = 0$

18. a) Show that $\sqrt{\frac{3^{x+1}-3^{x}}{3^{x-1}} + 3}$ is equal to 3

    b) Hence solve $\sqrt{\frac{3^{x+1}-3^{x}}{3^{x-1}} + 3} = \left(\frac{1}{3}\right)^{x-2}$

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1a. 2258	1b. 2259	2a. 225B	2b. 225C	2c. 225D	2d. 225F
2e. 225G	2f. 225H	3a. 225J	3b. 225K	3c. 225M	3d. 225N
3e. 225P	4. 225Q	5. 225R	6. 225S	7. 225T	8a. 225V
8b. 225W	8c. 225X	8d. 225Y	8e. 225Z	8f. 2262	9a. 2263
9b. 2264	9c. 2265	10a. 2266	10b. 2267	11. 2268	12a. 2269
12b. 226B	12c. 226C	13. 226D	14. 226F	15. 226G	16. 226H
17a. 226J	17b. 226K	17c. 226M	17d. 226N	17e. 226P	18. 226Q

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